

Algebraic contributions to equivariant Iwasawa theory

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von Irene Lau

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Erstgutachter: Prof. Dr. Jürgen Ritter

Zweitgutachter: Prof. Dr. Otmar Venjakob

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Introduction

The word ‘contributions’ of the title implies a full description of the algebraic structure of the semisimple algebra $\mathcal{Q}G = \text{Quot}(\Lambda G)$, where ΛG , for a fixed odd prime number l , is the Iwasawa algebra of the Galois group G of a pro- l Galois extension K/k of totally real number fields, with k/\mathbb{Q} finite and K containing the cyclotomic \mathbb{Z}_l -extension k_∞/k of finite index. It moreover implies a reduction of the main conjecture of equivariant Iwasawa theory to the pro- l case, its uniqueness statement included. The setting and issue of the paper will be discussed in more detail in the following paragraphs.

Iwasawa theory is a recent branch of number theory, named after the Japanese mathematician Kenkichi Iwasawa (1917-1998). One of Iwasawa’s aims was to examine the asymptotic behaviour of the class numbers of cyclotomic extensions. He conjectured a strong relationship between the so called Iwasawa module X , which is the projective limit of class groups, and Artin L -functions. This astonishing link between the arithmetic data X and the analytic data L was first proven by Mazur and Wiles ([21]) and in a more general setting by Wiles ([46]).

Nevertheless, this branch is still rigorously studied and many approaches have been done to generalize the settings and to construct a non-commutative Iwasawa theory.

The idea to construct an equivariant Iwasawa theory using the language of K -theory was initiated by Jürgen Ritter and Alfred Weiss in [30] and is still the object of their present research. They are about to prove a generalized main conjecture for certain one-dimensional l -adic Lie groups in the following setting.

Let l be an odd prime and K/k a Galois extension of totally real number fields with Galois group G such that k/\mathbb{Q} and K/k_∞ are finite. As usual, k_∞ denotes the cyclotomic \mathbb{Z}_l -extension of k .

Next, the Iwasawa algebra $\Lambda G = \mathbb{Z}_l[[G]]$ denotes the completed group ring of G over \mathbb{Z}_l and $\mathcal{Q}G = \text{Quot}(\mathbb{Z}_l[[G]])$ is its total ring of fractions with respect to all central non-zero divisors. $\mathcal{Q}G$ finds its way into non-commutative Iwasawa theory via the localization sequence of K -theory

$$\rightarrow K_1(\Lambda G) \rightarrow K_1(\mathcal{Q}G) \xrightarrow{\partial} K_0T(\Lambda G) \rightarrow$$

and a determinant map

$$\text{Det} : K_1(\mathcal{Q}G) \rightarrow \text{Hom}(R_l G, (\mathbb{Q}_l^c \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma_k)^\times).$$

As in the classical case of Iwasawa, i.e. $k = \mathbb{Q}$ and $K = \mathbb{Q}_\infty$, Ritter and Weiss try to link a K -theoretic substitute $\mathcal{U} \in K_0 T(\Lambda G)$ of the Iwasawa module X to the Iwasawa L -function which is derived from the S -truncated Artin L -function for a finite set S of places of k containing all archimedean ones and those which ramify in K . This Iwasawa L -function lies in the upper Hom-group. Here, as usual, $X = G(M/K)$ with M denoting the maximal abelian l -extension of K which is unramified outside S .

With this, the main conjecture of equivariant Iwasawa theory says

$$\exists! \Theta \in K_1(\mathcal{Q}G) : \text{Det}(\Theta) = L \ \& \ \partial(\Theta) = \mathcal{U}.$$

In the classical setting, this is exactly the classical Iwasawa Main Conjecture mentioned above. The uniqueness of Θ would follow from a conjecture by Suslin being true. This conjecture applied to our situation means that $SK_1(\mathcal{Q}G) = \ker(\text{Det}) = 1$ (we suppress here the condition on the cohomological dimension of the centres which we show to be fulfilled in the last chapter).

So far, a complete proof of the existence of such a Θ does not exist. Nevertheless, Hara ([11], [12]), Kakde ([14]), Kato ([16]), and Ritter and Weiss ([36]) have shown the conjecture in some special cases, provided that the Iwasawa μ -invariant vanishes. Ritter and Weiss have also reduced the existence of Θ in the general pro- l case to the integrality of a so-called logarithmic pseudomeasure. Moreover, they have reduced the case of general profinite groups G to pro- l elementary Galois groups.

Recently (compare [38]), Ritter and Weiss have reduced the main conjecture of equivariant Iwasawa theory to a purely algebraic statement, the so-called Möbius-Wall congruence, which generalizes the torsion congruence (1.2) appearing in Chapter 1. The Möbius-Wall congruence itself, however, has only been verified in special cases yet, and it is therefore too early to speculate whether it, as (1.2), generalizes to the components $T(e_i Q_\wedge G)$ for pro- l elementary Galois groups G (see Chapter 2). If so, then, up to its uniqueness statement, the main conjecture of equivariant Iwasawa theory is true whenever Iwasawa's μ -invariant vanishes.

A different approach to formulate a main conjecture of non-commutative Iwasawa theory, which applies to more general situations K/k , was made by Coates, Fukaya, Kato, Sujatha and Venjakob in [2] and later on by Fukaya and Kato in [8] and Kato in [15].

The questions we are dealing with in this paper have all arisen from the aim to prove the main conjecture of equivariant Iwasawa theory. Yet, it might be interesting to

free the studied objects from this background. The algebra $\mathcal{Q}G$ in particular is an interesting algebraic object in itself and the results achieved here may be interesting also from a purely algebraic point of view because they show that properties of group algebras of finite groups do generally not persist when passing to projective limits of such. And, of course, the study of the Suslin conjecture in Chapters 4 and 5 connects this paper to work on relative K -theory and the Tannaka-Artin problem in Suslin's refined version.

In the first chapter, we formulate the main conjecture of equivariant Iwasawa theory in detail after having given an overview on classical Iwasawa theory. Then, we recollect the proof of Ritter and Weiss that, for pro- l groups, the main conjecture of equivariant Iwasawa theory, up to uniqueness of Θ , is equivalent to the above mentioned integrality property, provided that Iwasawa's μ -invariant vanishes.

In the second chapter, we reduce the proof of the main conjecture of equivariant Iwasawa theory, up to uniqueness, to the case of a pro- l Galois group G extended by finite unramified extensions N of \mathbb{Q}_l , i.e. we consider the algebra $N \otimes_{\mathbb{Q}_l} \mathcal{Q}G$ instead of $\mathcal{Q}G$. More precisely, we show that the integrality of the logarithmic pseudomeasure is unaffected by extension of the coefficients to an unramified extension. Therefore, the equivariant main conjecture, up to uniqueness, is true for arbitrary profinite Galois groups iff the logarithmic pseudomeasure extended by finite unramified extensions of \mathbb{Q}_l is integral for pro- l Galois groups, provided that Iwasawa's μ -invariant vanishes.

In the third chapter, we resolve the structure of the algebra $\mathcal{Q}G$ for pro- l groups G . Here, we restrict ourselves to the case of pro- l groups because, as we have seen in Chapter 2, this is the crucial case. Roquette (see [39]) showed that for finite l -groups H , the group algebra $F[H]$ over a field F of characteristic 0 splits into full matrix rings over fields, i.e. that skew fields do not appear in the Wedderburn components of $F[H]$. Although $\mathcal{Q}G = \text{Quot}(\mathbb{Z}_l[[G]])$ is not a group algebra, one might expect that there do not occur Schur indices in its structure because G is as pro- l group the projective limit of finite l -groups. However, this is not the case: As we show in Chapter 3, non-trivial Schur indices appear, but the occurring skew fields are all cyclic. We resolve the algebraic structure of the Wedderburn components of $\mathcal{Q}G$ completely. Moreover, we give an example in which such a non-trivial case appears as Galois group in the equivariant Iwasawa setting.

In the fourth chapter, we reduce the Suslin conjecture for our Iwasawa algebra $\mathcal{Q}G$ for profinite Galois groups G to the conjecture for $N \otimes_{\mathbb{Q}_l} \mathcal{Q}U$ for pro- l groups U and finite unramified extensions N of \mathbb{Q}_l . Therefore, the proof of the main conjecture of equivariant Iwasawa theory is completely reduced to pro- l groups provided that the studied objects are unaffected by passing to finite unramified extensions of \mathbb{Q}_l .

In the last chapter, we consider fields of type $\mathbb{Q}_l(\zeta) \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma$ with ζ a primitive l -power

root of unity and $\Gamma \cong \mathbb{Z}_l$. We compute the cohomological dimensions of these fields and their \mathfrak{p} -adic completions for primes \mathfrak{p} of height 1. Furthermore, we use this to show that the Suslin conjecture is true for the completed $\mathcal{Q}_\wedge G$ if the completion is not with respect to the prime above l . For the completion with respect to l , we are at least able to compute the completion of the skew fields underlying the simple components of $\mathcal{Q}G$ and their residue skew fields. Although these last results seem to be less naturally connected to the previous chapters, we hope that they might be useful for future work on Suslin's conjecture.

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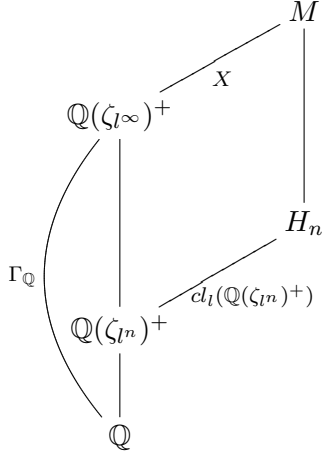
I am grateful to my husband and my parents for their non-mathematical, but just as important, support.

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1 Recollections

1.1 Iwasawa theory

1.1.1 Classical Iwasawa theory



Initially, Iwasawa was interested in the study of the structure of the class groups $cl(\mathbb{Q}(\zeta_{l^n}))$ of cyclotomic fields $\mathbb{Q}(\zeta_{l^n})$, or more explicitly of the class group of the plus parts $\mathbb{Q}(\zeta_{l^n})^+$. From now on, l will always be an odd prime number.

He first found an interesting property of the direct limit $\varinjlim \mathbb{Q}(\zeta_{l^n})^+ =: \mathbb{Q}(\zeta_{l^\infty})^+$ of these plus parts, namely that the Galois group $G(\mathbb{Q}(\zeta_{l^\infty})^+/\mathbb{Q})$ is isomorphic to the group $\varprojlim G(\mathbb{Q}(\zeta_{l^n})^+/\mathbb{Q}) \cong \varprojlim (\mathbb{Z}/l^n\mathbb{Z}) = \mathbb{Z}_l$ of l -adic integers. This Galois group will be denoted by $\Gamma_{\mathbb{Q}}$ in the sequel. For his study of class groups, Galois theoretical arguments turned out to be useful because, by class

field theory, the class group of a field F can be read as the Galois group of the maximal abelian unramified field extension of F ; and analogously the l -part $cl_l(F)$ of the class group corresponds to the maximal abelian unramified l -extension of F . In our situation, we denote the maximal abelian unramified l -extension of $\mathbb{Q}(\zeta_{l^n})^+$ by H_n and the maximal abelian unramified l -extension of $\mathbb{Q}(\zeta_{l^\infty})^+$ by M . Then, $X := G(M/\mathbb{Q}_l(\zeta_{l^\infty}))$ is isomorphic to the projective limit of the $cl_l(\mathbb{Q}(\zeta_{l^n})^+)$ and thus carries information of the asymptotic behaviour of the class groups.

Observe that M is Galois over \mathbb{Q} because each H_n is Galois over \mathbb{Q} as the H_n are maximal. We set $\mathfrak{G} := G(M/\mathbb{Q})$ and see that $\Gamma_{\mathbb{Q}} \cong \mathfrak{G}/X$. Thus, X can be interpreted as a $\Gamma_{\mathbb{Q}}$ -module. Furthermore, we set $\Lambda := \mathbb{Z}_l[[T]] \cong \mathbb{Z}_l[[\Gamma_{\mathbb{Q}}]]$. Then, it can be seen that X is a Λ -module. Together with this latter structure, X is called the *Iwasawa module*. As Λ -module, X is finitely generated and torsion and therefore pseudo-isomorphic to a finite direct sum $\bigoplus_r \Lambda/(l^{m_r}) \oplus \bigoplus_s \Lambda/(f_s^{n_s})$ with f_s not necessarily distinct irreducible polynomials. In this case, or more general for cyclotomic \mathbb{Z}_l -extensions of abelian number fields, $\sum_r m_r =: \mu$ vanishes by a result

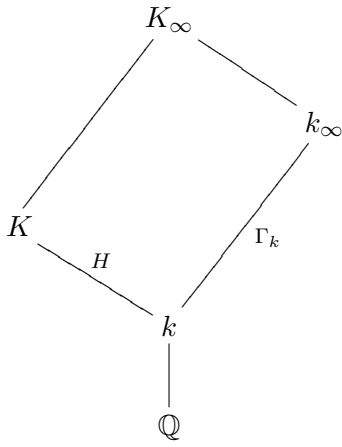
of Ferrero-Washington (see [6]). Thus, we obtain

$$X \sim \bigoplus_s \Lambda / (f_s^{n_s}).$$

This gives rise to the definition of the characteristic polynomial of X ,

$$ch(X) := \prod_s f_s^{n_s} \in \Lambda,$$

which carries much information about X .



This situation was then extended to the more general case of a totally real finite extension k of \mathbb{Q} . Let k_∞ be the cyclotomic \mathbb{Z}_l -extension of k , thus $G(k_\infty/k) = \Gamma_k \cong \mathbb{Z}_l$, and K a finite totally real abelian extension of k with Galois group H and the order of H prime to l . This implies that $k_\infty \cap K = k$. We set $K_\infty = k_\infty K$ and denote its Galois group over k by $G \cong H \times \Gamma_k$. Since H is of order prime to l , the completed group ring $\Lambda G := \mathbb{Z}_l[[G]] = \mathbb{Z}_l[H][[\Gamma_k]]$ splits into the direct sum

$$\Lambda G = \bigoplus_i' \mathbb{Z}_l[\chi_i][[\Gamma_k]] \cong \bigoplus_i' \mathbb{Z}_l[\chi_i][[T]],$$

where χ_i runs through the irreducible \mathbb{Q}_l^\times -characters of H modulo the action of $G(\mathbb{Q}_l^\times/\mathbb{Q}_l)$.

As above, we set X the Galois group of the maximal abelian unramified l -extension M of K_∞ ; recall that M is Galois over k . X is still a finitely generated ΛG -torsion module. Moreover, the above splitting of ΛG induces a decomposition of X into components corresponding to the irreducible \mathbb{Q}_l^\times -characters of H :

$$X \cong \bigoplus_i' X_{\chi_i},$$

where the are finitely generated $\mathbb{Z}_l[\chi_i][[T]]$ -torsion modules. The structure of those X_{χ_i} yield, as in the above case, characteristic polynomials $f_{\chi_i} \in \mathbb{Z}_l[\chi_i][[T]]$ corresponding to the X_{χ_i} .

For the moment, we turn to Artin L -functions which play an important role in the conjecture of Iwasawa. The Artin L -function is a generalization of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $s \in \mathbb{C}$, to a finite Galois extension of number fields. First, this zeta function can be defined in the slightly more general context of a number field k with $[k : \mathbb{Q}] < \infty$. We obtain the Dedekind zeta function

$$\zeta_k(s) = \sum_{\mathfrak{a}} \frac{1}{N_{k/\mathbb{Q}}(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N_{k/\mathbb{Q}}(\mathfrak{p})^{-s}},$$

where \mathfrak{a} denotes an integral ideal and \mathfrak{p} a prime ideal of k and $N_{k/\mathbb{Q}}$ is the ideal norm.

Next, let K/k be a finite Galois extension of number fields with Galois group G . We choose a prime \mathfrak{p} of k and a prime \mathfrak{P} of K above \mathfrak{p} . $G_{\mathfrak{P}}$ denotes the decomposition group and $I_{\mathfrak{P}}$ the inertia group of \mathfrak{P} over \mathfrak{p} . We set $\kappa(\mathfrak{P}) := \mathfrak{o}_K/\mathfrak{P}$ resp. $\kappa(\mathfrak{p}) := \mathfrak{o}_k/\mathfrak{p}$ for the residue fields. Via the canonical isomorphism $G_{\mathfrak{P}}/I_{\mathfrak{P}} \cong G(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$, the factor group $G_{\mathfrak{P}}/I_{\mathfrak{P}}$ can be read as generated by the Frobenius automorphism $\varphi_{\mathfrak{P}}$ which sends $x \in \kappa(\mathfrak{P})$ to x^q with $q = N(\mathfrak{p})$. For the representation module V of a representation D of $G(K/k)$, this $\varphi_{\mathfrak{P}}$ is an endomorphism of the submodule $V^{I_{\mathfrak{P}}}$ fixed by $I_{\mathfrak{P}}$.

Then, the Artin L -function is defined as

$$L_{K/k}(\chi, s) = \prod_{\mathfrak{p}} \frac{1}{\det(1 - \varphi_{\mathfrak{P}} N_{k/\mathbb{Q}}(\mathfrak{p})^{-s} | V^{I_{\mathfrak{P}}})},$$

where the product runs through the prime ideals of k and χ is the character corresponding to the representation D . This definition does not depend on the special \mathfrak{P} chosen to lie above \mathfrak{p} .

Observe that, for the trivial character, this is exactly the Dedekind zeta function for the field k .

The Artin L -function still hides plenty of its properties. For example, the Artin conjecture that $L_{K/k}(\chi, s)$ is an entire function for every irreducible nontrivial character is still open. Its values at the negative integers are also unknown. At least, these values are known to be algebraic over \mathbb{Q} and for not totally real fields k they are 0.

For further information on Artin L -functions, we refer to [24].

In [17], Kubota and Leopoldt first constructed an l -adic analogon, the l -adic Artin L -function. χ is now a \mathbb{Q}_l^c -character of $G(K/k)$ and we set

$$L_{K/k,l}(\chi, 1 - n) = L_{K/k}(\chi, 1 - n) \prod_{\mathfrak{p}|l} \det(1 - \varphi_{\mathfrak{P}} N(\mathfrak{p})^{n-1} | V^{I_{\mathfrak{P}}})$$

for all $n \in \mathbb{N}$ with $1 - n \equiv 1 \pmod{l-1}$. For general $s \in \mathbb{Z}_l$ the values $L_{K/k,p}(\chi, s)$ are obtained by l -adic interpolation. This yields a meromorphic function with values in \mathbb{Q}_l^c since the Artin L -function is algebraic over \mathbb{Q} at the negative integers.

Obviously, for the search for values at the negative integers studying the Artin L -function is as good as its l -adic analogon.

Now, consider a totally real finite extension k of \mathbb{Q} and the cyclotomic \mathbb{Z}_l -extension k_{∞} of k , thus $G(k_{\infty}/k) = \Gamma_k \cong \mathbb{Z}_l$. Let K be a finite totally real Galois extension

of k with Galois group H and $k_\infty \cap K = k$, denote $K_\infty = k_\infty K$. For the Galois extension K_∞/k , the l -adic Artin L -function can be defined as above for all \mathbb{Q}_l^c -characters of $G := G(K_\infty/k)$ with open kernel; more generally, for every finite set S of places of k , the S -truncated L -function is

$$L_{K_\infty/k, l, S}(\chi, 1-n) = L_{K_\infty/k, l}(\chi, 1-n) \prod_{\mathfrak{p} \in S, \mathfrak{p} \nmid l\infty} \det(1 - \varphi_{\mathfrak{p}} N(\mathfrak{p})^{n-1} | V^{I_{\mathfrak{p}}}).$$

Cassou-Noguès ([1]) and Deligne and Ribet ([3]) showed that this l -adic L -function is essentially an l -adic power series:

$$L_{K_\infty/k, l, S}(s, \chi) = \frac{G_{\chi, S}(u^s - 1)}{H_\chi(u^s - 1)}$$

with γ_k a topological generator of Γ_k and $u \in 1 + l\mathbb{Z}_l$ such that $\zeta^{\gamma_k} = \zeta^u$ for all l -power roots of unity ζ . Then $G_{\chi, S}(T) \in \mathbb{Q}_l^c \otimes_{\mathbb{Q}_l} \text{Quot}(\mathbb{Z}_l[[T]])$ and

$$H_\chi(T) = \begin{cases} \chi(\gamma_k)(1+T) - 1 & \text{if } H \subseteq \ker \chi, \\ 1 & \text{else.} \end{cases}$$

Actually, this $G_{\chi, S}$ is an l -adic power series in $\mathbb{Z}_l[\chi][[T]]$ (compare [32, p. 571]).

We have come up with two special power series, namely the characteristical polynomials of Iwasawa modules on the one hand and the Artin L -functions on the other hand. Iwasawa first saw that there could be a link between those objects and conjectured that they were equal up to units:

Iwasawa Main Conjecture *Let k be a totally real finite extension of \mathbb{Q} and k_∞ the cyclotomic \mathbb{Z}_l -extension of k with Galois group Γ_k . Set K a finite totally real abelian l -prime extension of k with Galois group H and $K_\infty = k_\infty K$. Finally, let K_∞/k be abelian with Galois group $G \cong H \times \Gamma_k$. Then*

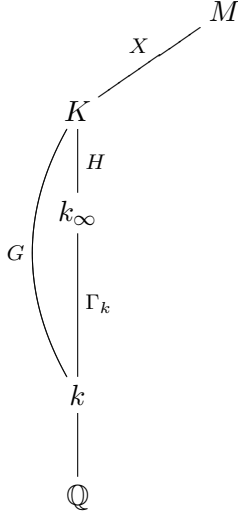
$$\text{ch}(X_{\chi_i}) \doteq G_{\chi_i}(T),$$

where χ_i runs through the irreducible \mathbb{Q}_l^c -characters of H and \doteq means ‘equal up to units’.

This conjecture was proven by Mazur and Wiles in [21] for $k = \mathbb{Q}$ and trivial H , and Wiles extended their results in [46] to the case formulated above.

The generalization from trivial to nontrivial abelian H of order prime to l has been sketched earlier in this chapter. But it is hard to skip the assumption that G is abelian or even that H is of order prime to l because in these cases, the ring $\mathbb{Z}_l[[G]]$ becomes fairly complicated and we do not have much information about the structure of its modules.

1.1.2 Equivariant Iwasawa theory



Ritter and Weiss suggest a conjectural generalization of the classical Iwasawa Main Conjecture via K -theoretic methods. In the sequel, we will state their conjecture and sketch the reduction of this generalized main conjecture to an integrality property in the case of pro- l Galois groups.

Again, we fix a prime number $l \neq 2$ and a totally real finite field extension k/\mathbb{Q} . Then, we denote the cyclotomic \mathbb{Z}_l -extension of k by k_∞ and let $K \supseteq k_\infty$ be a totally real Galois extension of k with Galois group G . Assume that $H := G(K/k_\infty)$ is finite. M denotes the maximal abelian l -extension of K unramified outside S , where S is a finite set of places of k which contains all infinite places and all places ramified in K/k . Thus S contains all places above l . Again, M is Galois over k .

The Iwasawa module X is a finitely generated ΛG -torsion module by [25, 11.3.1, 10.3.25, 11.3.2]. Let $K_0T(\Lambda G)$ denote the Grothendieck group of the category of finitely generated torsion ΛG -modules of finite projective dimension. It is not possible to view X in $K_0T(\Lambda G)$ directly because X does not have finite projective dimension in general. Thus, we construct an element $\bar{U} \in K_0T(\Lambda G)$ carrying all information about X . To do so, we first apply the translation functor to the short exact sequence of Galois groups

$$X \hookrightarrow G(M/k) \twoheadrightarrow G$$

and obtain a short exact sequence of finitely generated ΛG -modules

$$X \hookrightarrow Y \twoheadrightarrow \Delta G$$

with ΔG being the kernel of the augmentation map $\mathbb{Z}_l[[G]] \rightarrow \mathbb{Z}_l$. For finite groups G , this functor is explained in [29, pp. 154-155] and the extension to profinite groups G can be found in [30, 4.] as well as the proof that $\text{pd}(Y) \leq 1$. We obtain the commutative diagram

$$\begin{array}{ccccc}
 & \Lambda G & \xlongequal{\quad} & \Lambda G & \\
 & \downarrow \Psi & & \downarrow \psi & \\
 X \hookrightarrow & Y & \twoheadrightarrow & \Delta G & \\
 \parallel & \downarrow & & \downarrow & \\
 X \hookrightarrow & \text{coker } \Psi & \twoheadrightarrow & \text{coker } \psi &
 \end{array}$$

with injective maps ψ, Ψ by [30, Prop 4.5]; moreover, ψ and Ψ have torsion cokernel. As $\text{pd}(Y) \leq 1$, also $\text{pd}(\text{coker } \Psi) \leq 1$. We conclude $[\text{coker } \Psi] \in K_0T(\Lambda G)$ but this class depends on the choices made. To drop those dependencies, we extend the diagram with respect to the augmentation map and get the commutative diagram

$$\begin{array}{ccccccc}
 & & \Lambda G & \xlongequal{\quad} & \Lambda G & & \\
 & & \downarrow \Psi & & \downarrow \tilde{\psi} & & \\
 X^{\subset} & \longrightarrow & Y & \longrightarrow & \Lambda G & \twoheadrightarrow & \mathbb{Z}_l \\
 \parallel & & \downarrow & & \downarrow & & \parallel \\
 X^{\subset} & \longrightarrow & \text{coker } \Psi & \longrightarrow & \text{coker } \tilde{\psi} & \twoheadrightarrow & \mathbb{Z}_l
 \end{array}$$

Finally,

$$\mathfrak{U} := [\text{coker } \Psi] - [\text{coker } \tilde{\psi}] \in K_0T(\Lambda G)$$

can be seen as a substitute of X independent of all choices. \mathfrak{U} only depends on S but not on the choice of γ_k . For details of the constructions see [30], [31] and [32].

Our aim is now to relate this \mathfrak{U} to the l -adic Artin L -function $L_{K/k,l,S}$. We have already seen that

$$L_{K/k,l,S}(s, \chi) = \frac{G_{\chi,S}(u^s - 1)}{H_{\chi}(u^s - 1)}$$

with $u \in 1 + l\mathbb{Z}_l$ such that $\zeta^{\gamma_k} = \zeta^u$ for all l -power roots of unity ζ and

$$H_{\chi}(T) = \begin{cases} \chi(\gamma_k)(1 + T) - 1 & \text{if } H \subseteq \ker \chi, \\ 1 & \text{else.} \end{cases}$$

We set $R_l G$ the \mathbb{Z} -span of all irreducible \mathbb{Q}_l^{\subset} -characters of G with open kernel. Then $\text{Hom}^N(R_l G, \cdot)$ denotes the group of all homomorphisms $f \in \text{Hom}(R_l G, \cdot)$ with the following properties:

- $f(\chi^{\sigma}) = f(\chi)^{\sigma}$ for $\sigma \in G(\mathbb{Q}_l^{\subset}/N)$,
- $f(\chi \otimes \rho) = \rho^{\sharp}(f(\chi))$ for all characters of type W , i.e. abelian characters acting trivially on H . Recall that the W -twist on $\mathcal{Q}^{\subset} \Gamma_k := \mathbb{Q}_l^{\subset} \otimes_{\mathbb{Q}_l} \mathcal{Q} \Gamma_k$ is induced by $\rho^{\sharp}(\gamma_k) = \rho(\gamma_k)\gamma_k$, where $\mathcal{Q} \Gamma_k := \text{Quot}(\Lambda \Gamma_k)$ denotes the quotient field of $\Lambda \Gamma_k$.

For $N = \mathbb{Q}_l$ we define $\text{Hom}^*(R_l G, \cdot) := \text{Hom}^{\mathbb{Q}_l}(R_l G, \cdot)$.

To view the L -function in $\text{Hom}^*(R_l G, (\mathcal{Q}^{\subset} \Gamma_k)^{\times})$, we set

$$L_{K/k}(\chi) := L_{K/k,l,S}(\chi) := \frac{G_{\chi,S}(\gamma_k - 1)}{H_{\chi}(\gamma_k - 1)}$$

and call $L_{K/k}(\chi)$ the *Iwasawa L -function*. It does not depend on the choice of γ_k , see [32, Prop. 11]. Furthermore, we will see in Lemma 1.3 that the Iwasawa L -function is compatible with restriction and deflation of the group G .

Now, we define the determinant map

$$\text{Det} : K_1(\mathcal{Q}G) \rightarrow \text{Hom}^*(R_l G, (\mathcal{Q}^c \Gamma_k)^\times),$$

where $\mathcal{Q}G := \text{Quot}(\Lambda G)$ denotes the total ring of fractions of ΛG with respect to all central non-zero divisors. Later, we will see that $\mathcal{Q}G$ is a semisimple algebra whose Wedderburn components have centres contained in $\mathcal{Q}^c \Gamma_k$. Moreover, these Wedderburn components are in 1-1 correspondence with the irreducible \mathbb{Q}_l^c -characters χ of G with open kernel, modulo the Galois action of $G(\mathbb{Q}_l^c/\mathbb{Q}_l)$ and modulo W -twists. Now, the map ‘Det’ is induced by taking reduced norms on the Wedderburn components. For this, we recall from [32] that we have an isomorphism

$$Z(\mathcal{Q}G)^\times \cong \text{Hom}^*(R_l G, (\mathcal{Q}^c \Gamma_k)^\times),$$

where $Z(\mathcal{Q}G)$ denotes the centre of $\mathcal{Q}G$. Furthermore, $Z(\mathcal{Q}G)^\times = K_1(Z(\mathcal{Q}G))$. Then, Det is the composition of the reduced norm (on the Wedderburn components) with this isomorphism. In particular, Det is injective resp. surjective iff the reduced norm is.

The following diagram, with top row the localization sequence of K -theory, shows the relation between the objects defined so far.

$$\begin{array}{ccccccc} & & & \mathcal{U} & & & \\ & & & \cap & & & \\ K_1(\Lambda G) & \longrightarrow & K_1(\mathcal{Q}G) & \xrightarrow{\partial} & K_0 T(\Lambda G) & \longrightarrow & K_0(\Lambda G) \longrightarrow K_0(\mathcal{Q}G) \\ & & \text{Det} \downarrow & & & & \\ & & L_{K/k} \in \text{Hom}^*(R_l G, (\mathcal{Q}^c \Gamma_k)^\times) & & & & \end{array}$$

We are finally ready to state the main conjecture of equivariant Iwasawa theory:

Equivariant main conjecture *There exists a unique element $\Theta \in K_1(\mathcal{Q}G)$ with $\text{Det}(\Theta) = L_{K/k}$. This Θ in addition satisfies $\partial(\Theta) = \mathcal{U}$.*

From now on, we will refer to the equivariant main conjecture as (MC).

The uniqueness of this conjecture would be a consequence of a conjecture of Suslin which says that the kernel $SK_1(\mathcal{Q}G)$ of Det is trivial. We will go into more detail about this in Section 1.3.

So far, a complete proof of the existence of such a Θ does not exist. In the sequel, we sketch how Ritter and Weiss have reduced the general pro- l case to the proof of an integrality property.

First, the abelian case is true:

Proposition 1.1 *Assume that $\mu = 0$ and G is abelian. Then the equivariant main conjecture is true.*

Proof: This is shown in [31]. □

Next, we examine the situation of a not necessarily abelian pro- l group G . Here, the conjecture is reduced to a conjectural property of $L_{K/k}$.

Definition 1.1 *Write $\Lambda_\bullet G := (\mathbb{Z}_l[[G]])_{(l)}$ for the localization of ΛG at the prime ideal (l) of its centre and $\Lambda_\wedge G$ for the completion of $\Lambda_\bullet G$ with respect to the (l) -adic topology.*

Proposition 1.2 *Assume that $\mu = 0$ and G is a pro- l group. Then the following statements are equivalent to the equivariant main conjecture up to uniqueness of Θ :*

- $L_{K/k} \in \text{Det}(K_1(\Lambda_\bullet G))$.
- $L_{K/k} \in \text{Det}(K_1(\Lambda_\wedge G))$.

Proof: For the proof see [34] and [33]. □

Proposition 1.1 actually becomes a corollary to Proposition 1.2 when exploiting Serre's pseudomeasure $\lambda_{K/k} \in (\Lambda_\bullet G)^\times$ which satisfies $\text{Det}(\lambda_{K/k}) = L_{K/k}$. Here, we need that $\mu = 0$ to guarantee that $\lambda_{K/k}$ is a unit in $\Lambda_\bullet G$. For details see [41]. Because we have $K_1(\Lambda_\bullet G) = (\Lambda_\bullet G)^\times$ in the abelian case, this is exactly the assertion of Proposition 1.2 and the pseudomeasure can be read as the conjectured Θ .

The main tool for the proof of Proposition 1.2 is to introduce an integral logarithm \mathbf{L} generalizing the group logarithm for finite l -groups constructed by M.J. Taylor and R. Oliver (see [44]).

Having in mind the reduction of the case of general profinite groups to pro- l groups, we state the following in a more general situation. Let N/\mathbb{Q}_l be a finite unramified extension. Denote by \mathfrak{o} the integral closure of \mathbb{Z}_l in N and define $\Lambda^\circ G := \mathfrak{o} \otimes_{\mathbb{Z}_l} \Lambda G$. Analogously, we set $\mathcal{Q}^N G := N \otimes_{\mathbb{Q}_l} \mathcal{Q} G$. To give the definition of \mathbf{L} , we need some notation first. The map Ψ on $\Lambda^\circ G$ for \mathfrak{o} a finite extension of \mathbb{Z}_l is defined to be the \mathfrak{o} -linear map sending g to g^l for $g \in G$. Thus, by identifying $\Lambda \Gamma_k$ with $\mathbb{Z}_l[[T]]$ via $\gamma_k \leftrightarrow (1+T)$, this Ψ sends T to $(1+T)^l - 1$. Note that Ψ is a ring homomorphism for abelian G . Furthermore, we need the Adams operation $\psi_l : R_l G \rightarrow R_l G$ induced by $(\psi_l(\chi))(g) = \chi(g^l)$ for $g \in G$.

Now, we set $\Lambda^c \Gamma_k := \mathbb{Z}_l^c \otimes_{\mathbb{Z}_l} \Lambda \Gamma_k$. Let Fr denote the automorphism of $\mathcal{Q}^N G$ induced by the Frobenius automorphism of N/\mathbb{Q}_l . (Here, we need N/\mathbb{Q}_l to be unramified.) Fr acts on $\text{Hom}^N(R_l G, (\Lambda^c \Gamma_k)^\times)$ by $f^{\text{Fr}}(\chi) = f(\chi^{\varphi^{-1}})^\varphi$, where φ is any lift of Fr to

$G(\mathbb{Q}_l^c/\mathbb{Q}_l)$. Finally, we define

$$\begin{aligned} \text{HOM}^N(R_l G, (\Lambda^c \Gamma_k)^\times) \\ := \{f \in \text{Hom}^N(R_l G, (\Lambda^c \Gamma_k)^\times) : f(\chi)^l \equiv \Psi(f^{\text{Fr}}(\psi_l \chi)) \bmod (l\Lambda^c \Gamma_k)\} \end{aligned}$$

and $\text{HOM}(R_l G, (\Lambda^c \Gamma_k)^\times) := \text{HOM}^{\mathbb{Q}_l}(R_l G, (\Lambda^c \Gamma_k)^\times)$ for $N = \mathbb{Q}_l$.

Lemma 1.1 *The homomorphism $\mathbf{L} : \text{HOM}^N(R_l G, (\Lambda_\wedge^c \Gamma_k)^\times) \rightarrow \text{Hom}^N(R_l G, \mathcal{Q}_\wedge^c \Gamma_k)$ defined by*

$$f \mapsto \left[\chi \mapsto \frac{1}{l} \log \frac{f(\chi)^l}{\Psi(f^{\text{Fr}}(\psi_l \chi))} \right]$$

has image in $\text{Hom}^N(R_l G, \Lambda_\wedge^c \Gamma_k)$.

Proof: This is [34, Lem 7]. The \wedge -index is needed in order to ensure the convergence of the log-series. \square

Proposition 1.3 *For the above determinant map Det , we obtain*

$$\text{Det}(K_1(\Lambda^\circ G)) \subseteq \text{HOM}^N(R_l G, (\Lambda^c \Gamma_k)^\times).$$

Proof: This can be found in [34, Thm 8]. \square

We also define $T(A) := A/[A, A]$ for a ring A , with $[A, A]$ the additive subgroup of A generated by all $ab - ba$ for $a, b \in A$; observe that $T(\Lambda^\circ G) \subseteq T(\mathcal{Q}_\wedge^N G)$.

Definition 1.2 *Let G be a pro- l group. Define the homomorphism \mathbb{L} by means of the commutative square*

$$\begin{array}{ccc} K_1(\Lambda^\circ G) & \xrightarrow{\quad \mathbb{L} \quad} & T(\mathcal{Q}_\wedge^N G) \\ \downarrow \text{Det} & & \cong \downarrow \text{Tr} \\ \text{HOM}^N(R_l G, (\Lambda_\wedge^c \Gamma_k)^\times) & \xrightarrow{\quad \mathbf{L} \quad} & \text{Hom}^N(R_l G, \mathcal{Q}_\wedge^c \Gamma_k) \end{array} \quad (1.1)$$

where the isomorphism Tr is induced by taking reduced traces on the Wedderburn components of $\mathcal{Q}_\wedge^N G$.

Proposition 1.4 *Let G be a pro- l group.*

- *In the commutative diagram (1.1), we obtain $\mathbb{L}(K_1(\Lambda^\circ G)) \subseteq T(\Lambda^\circ G)$.*
- *If S is sufficiently large and if $\mu = 0$, then $L_{K/k} \in \text{HOM}(R_l G, (\Lambda_\wedge^c \Gamma_k)^\times)$.*

Proof: See [34] for a proof. \square

For the rest of the chapter, we restrict ourselves to the case $N = \mathbb{Q}_l$. The existence of the pseudomeasure $\lambda_{K/k}$ in the abelian case together with diagram (1.1) suggests the following

Definition 1.3 Consider diagram (1.1) with $N = \mathbb{Q}_l$. The element $t_{K/k} \in T(\mathcal{Q}G)$ satisfying $\text{Tr}(t_{K/k}) = \mathbf{L}(L_{K/k})$ is called the logarithmic pseudomeasure.

Observe that because Tr is an isomorphism, $t_{K/k}$ always exists and is unique given S sufficiently large and $\mu = 0$, even if (MC) was false. Proposition 1.4 implies that $t_{K/k}$ has to be integral (i.e. $t_{K/k} \in T(\Lambda_\wedge G)$) if (MC) is true. It can be seen that the converse is true, too:

Theorem 1.1 Let G be a pro- l group, S sufficiently large and assume that $\mu = 0$. Then the equivariant main conjecture, up to uniqueness of Θ , is true if and only if $t_{K/k} \in T(\Lambda_\wedge G)$.

This is the main theorem of [37] together with [36]. It is the exact formulation of the reduction of (MC) to an integrality property, as mentioned at the beginning of this section.

In the rest of this section, we sketch the important steps of the proofs of this theorem. A main tool are the functorial properties of the objects of interest.

Definition 1.4 For G' an open subgroup of G with fixed field k' , resp. a factor group G/U by a finite normal subgroup, and for all $\chi' \in R_l G'$ we define the restriction and deflation maps

$$\text{res}_G^{G'} : \text{HOM}(R_l G, (\mathcal{Q}^c \Gamma_k)^\times) \rightarrow \text{HOM}(R_l G', (\mathcal{Q}^c \Gamma_{k'})^\times),$$

$$\text{defl}_G^{G'} : \text{HOM}(R_l G, (\mathcal{Q}^c \Gamma_k)^\times) \rightarrow \text{HOM}(R_l G', (\mathcal{Q}^c \Gamma_k)^\times)$$

by $(\text{res}_G^{G'} f)(\chi') = f(\text{ind}_{G'}^G(\chi'))$ and $(\text{defl}_G^{G'} f)(\chi') = f(\text{infl}_G^{G'}(\chi'))$.

The fact that induction and the Adams operation do not commute makes it necessary to introduce a modified restriction map Res in the T -world.

Definition 1.5 Let G be a pro- l group and keep the notations of Definition 1.4. We define the restriction map

$$\begin{aligned} \text{Res}_G^{G'} : \text{Hom}^*(R_l G, \mathcal{Q}_\wedge^c \Gamma_k) &\rightarrow \text{Hom}^*(R_l G', \mathcal{Q}_\wedge^c \Gamma_{k'}), \\ f &\mapsto \left[\chi \mapsto f(\text{ind}_{G'}^G \chi) + \sum_{r \geq 1} \frac{\Psi^r}{l^r} (f(\psi_l^{r-1} \chi)) \right], \end{aligned}$$

with $\chi = \psi_l(\text{ind}_{G'}^G \chi') - \text{ind}_{G'}^G(\psi_l \chi')$.

Via the trace isomorphism $\text{Tr} : T(\cdot) \rightarrow \text{Hom}^*(\cdot)$, we transport $\text{Res}_G^{G'}$ to a map, again called $\text{Res}_G^{G'}$, such that the diagram

$$\begin{array}{ccc} T(\mathcal{Q}_\wedge G) & \xrightarrow{\text{Tr}} & \text{Hom}^*(R_l G, \mathcal{Q}_\wedge^c \Gamma_k) \\ \downarrow \text{Res}_G^{G'} & & \downarrow \text{Res}_G^{G'} \\ T(\mathcal{Q}_\wedge G') & \xrightarrow{\text{Tr}'} & \text{Hom}^*(R_l G', \mathcal{Q}_\wedge^c \Gamma_{k'}) \end{array}$$

commutes.

Lemma 1.2 *The new restriction map $\text{Res}_G^{G'}$ makes the diagram*

$$\begin{array}{ccc} \text{HOM}(R_l G, (\Lambda_\wedge^c \Gamma_k)^\times) & \xrightarrow{\mathbf{L}} & \text{Hom}^*(R_l G, \mathcal{Q}_\wedge^c \Gamma_k) \\ \downarrow \text{res}_G^{G'} & & \downarrow \text{Res}_G^{G'} \\ \text{HOM}(R_l G', (\Lambda_\wedge^c \Gamma_{k'})^\times) & \xrightarrow{\mathbf{L}'} & \text{Hom}^*(R_l G', \mathcal{Q}_\wedge^c \Gamma_{k'}) \end{array}$$

commute. Here, \mathbf{L}' stands for the logarithm on the G' -level. Moreover, the following diagram commutes:

$$\begin{array}{ccccc} K_1(\Lambda_\wedge G) & \xrightarrow{\mathbf{L}} & T(\mathcal{Q}_\wedge G) & \xrightarrow{\text{Tr}} & \text{Hom}^*(R_l G, \mathcal{Q}_\wedge^c \Gamma_k) \\ \downarrow \text{res}_G^{G'} & & \downarrow \text{Res}_G^{G'} & & \downarrow \text{Res}_G^{G'} \\ K_1(\Lambda_\wedge G') & \xrightarrow{\mathbf{L}'} & T(\mathcal{Q}_\wedge G') & \xrightarrow{\text{Tr}'} & \text{Hom}^*(R_l G', \mathcal{Q}_\wedge^c \Gamma_{k'}) \end{array}$$

with again \mathbf{L}' resp. Tr' the logarithm resp. the trace map on the G' -level.

Proof: This is [36, Lem 1, Lem 2]. □

Lemma 1.3 *Let G' be an open subgroup of G with fixed field k' and U a finite normal subgroup of G . Then*

- (i) *Det commutes with restriction, deflation and induction on K_1 .*
- (ii) *Tr commutes with deflation and induction.*
- (iii) *\mathbf{L} commutes with deflation and induction.*
- (iv) *\mathbf{L} commutes with deflation and induction.*
- (v) *$\text{def}_G^{G/U}(L_{K/k}) = L_{K^U/k}$, $\text{res}_G^{G'}(L_{K/k}) = L_{K/k'}$.*
- (vi) *$\text{def}_G^{G/U}(t_{K/k}) = t_{K^U/k}$, $\text{Res}_G^{G'}(t_{K/k}) = t_{K/k'}$.*

Proof: The proofs can be found in the papers of Ritter and Weiss:

- (i) is stated in [32, Lem 9] and [34, Lem1].
- (ii) is stated in [37, proof of Lem 2.1] and [34, Lem 6].
- (iii) is stated in [36, p. 123] and [34, Lem 7].

(iv) is stated in [36, p. 123] and [34, p. 39].

(v) is [32, Prop 12].

(vi) is [37, Lem 2.1] resp. [36, Lem 2]. \square

Proposition 1.5 *Let G be a pro- l group. If $t_{K/k} \in T(\Lambda_\wedge G)$, i.e. the logarithmic pseudomeasure is integral, then $t_{K/k} \in \mathbb{L}(K_1(\Lambda_\wedge G))$. Moreover, there exists an element $y \in (\Lambda_\wedge G)^\times$ such that $\mathbb{L}(y) = t_{K/k}$ and $\text{defl}_G^{G^{\text{ab}}}(y) = \lambda_{K_{\text{ab}}/k}$, where K_{ab} is the fixed field of $[G, G]$ and $G^{\text{ab}} = G/[G, G]$.*

Proof: For a proof, we refer to [37, Prop 2.2]. \square

We next consider the case that G has an abelian subgroup G' of index l . We set $A := G/G' = \langle a \rangle$ and \mathcal{T}' to be the image of the A -trace map tr_A on $\Lambda_\wedge G'$, i.e.

$$\mathcal{T}' := \{\text{tr}_A y : y \in \Lambda_\wedge G'\} = \left\{ \sum_{i=0}^{l-1} y^{a^i} : y \in \Lambda_\wedge G' \right\}.$$

Moreover, denote by $\text{ver} : \Lambda_\wedge G \rightarrow \Lambda_\wedge G'$ the map induced by $G \rightarrow G^{\text{ab}} \rightarrow G'$.

Proposition 1.6 *Let G be a pro- l group with abelian subgroup G' of index l . Moreover, assume that S is sufficiently large and that $\mu = 0$. Then the following are equivalent.*

(i) $L_{K/k} \in \text{Det}(K_1(\Lambda_\wedge G))$,

(ii) $\text{ver}(\lambda_{K_{\text{ab}}/k}) \equiv \lambda_{K/k'} \pmod{\mathcal{T}'}$.

Proof: See [37, Prop 3.2] for a proof. \square

Proposition 1.7 *In the situation of Proposition 1.6, the congruence*

$$\text{ver}(\lambda_{K_{\text{ab}}/k}) \equiv \lambda_{K/k'} \pmod{\mathcal{T}'} \tag{1.2}$$

is true and therefore the main conjecture follows up to the uniqueness of Θ .

Proof: This is shown in [35]. \square

Finally, the torsion congruence (1.2) together with the functorial properties stated in Lemma 1.3 are the main ingredients of the proof of Theorem 1.1 (see [37]): It uses induction on the index of G over its centre. Via reduction and deflation, the L -function, the pseudomeasure for abelian subgroups and factor groups and the logarithmic pseudomeasure are put into relation and the congruence (2.2) finishes the proof.

1.2 The Iwasawa algebra $\mathcal{Q}G$

We here state some facts proven in [32] and introduce some notation. Recall that G splits: $G = H \rtimes \Gamma$ with $\Gamma = \langle \gamma \rangle \cong G(k_\infty/k)$. Thus, for a central subgroup $\Gamma^{l^m} =: \Gamma_0$ we get

$$\mathcal{Q}G = \bigoplus_{i=0}^{l^m-1} (\mathcal{Q}\Gamma_0)[H]\gamma^i.$$

This algebra is a finite dimensional $\mathcal{Q}\Gamma_0$ -algebra; in fact, it is a semisimple algebra, since the Jacobson radical is trivial. Now, let $\chi \in R_l G$ be an irreducible \mathbb{Q}_l^c -character of G with open kernel, i.e. there exists an integer \tilde{m} with $\tilde{\Gamma}_0 := \langle \gamma^{l^{\tilde{m}}} \rangle \subseteq \ker(\chi)$. Moreover, \tilde{m} can be chosen big enough such that $\tilde{\Gamma}_0$ is central in G . Thus, we assume m and \tilde{m} , resp. Γ_0 and $\tilde{\Gamma}_0$, to be equal. Note that it is sufficient to regard the finite set of irreducible characters of G/Γ_0 because, by inflation and W -twist, every irreducible $\chi \in R_l G$ with open kernel can be obtained from this set. Because G is an l -group with $l \neq 2$, this implies that χ has a representation over $\mathbb{Q}_l(\chi)$ by [39].

Furthermore, with η an absolutely irreducible constituent of $\text{res}_G^H(\chi)$, we define

$$St(\eta) := \{g \in G : \eta^g = \eta\}, \quad w_\chi := [G : St(\eta)]$$

and

$$e(\eta) := \frac{\eta(1)}{|H|} \sum_{h \in H} \eta(h^{-1})h.$$

Reading η as $\mathbb{Q}_l^c \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma_0 := \mathcal{Q}^c\Gamma_0$ -character, we obtain that $e(\eta)$ is a primitive central idempotent of the group ring $(\mathcal{Q}^c\Gamma_0)[H]$. Furthermore, Ritter and Weiss showed in [32] that

$$e_\chi := \sum_{\eta | \text{res}_G^H \chi} e(\eta)$$

is a central primitive idempotent in $\mathcal{Q}^c G$, that every central primitive idempotent is of the form e_χ and that two central primitive idempotents e_{χ_1} and e_{χ_2} coincide if and only if $\chi_1 = \chi_2 \otimes \rho$ for a character ρ of type W .

1.3 The Suslin conjecture

In this section, we give a brief overview of the conjecture of Suslin and the corresponding results so far.

Definition 1.6 (i) For a field F and a central simple F -algebra A of finite degree $[A : F]$, the group

$$SK_1(A) := \ker(\mathrm{nr}_{A/F})/[A^\times, A^\times]$$

is called the reduced Whitehead group of A .

(ii) For a semisimple algebra $A = \bigoplus_i A_i$ of finite degree with simple components A_i , we set

$$SK_1(A) := \bigoplus_i SK_1(A_i)$$

for the reduced Whitehead group of A .

The reduced norm $\mathrm{nr}_{A/F}$ on A induces a homomorphism on $K_1(A)$, which we will call reduced norm, too. We state the following results without proof.

Lemma 1.4 (i) Let A be a central simple F -algebra of finite degree. Then

$$SK_1(A) = \ker(\mathrm{nr}_{A/F} : K_1(A) \rightarrow K_1(F)).$$

(ii) Let $A \cong D_{n \times n}$ be the full matrix ring of finite degree over a skew field D . Then

$$SK_1(A) = SK_1(D).$$

(iii) For a field F , we have

$$SK_1(F) = 1.$$

For further details, see e.g. [5, Part III]. We are now ready to state the

Conjecture Let F be a field with cohomological dimension $\mathrm{cd}(F) \leq 3$ and A a central simple F -algebra of finite degree $[A : F]$. Then

$$SK_1(A) = 1.$$

In the following, we will refer to this as Suslin's conjecture, although this is not literally Suslin's formulation. But in the case of a field of cohomological dimension less or equal 3, this is exactly the statement of his conjecture. For details, we refer to [43].

Proposition 1.8 *If Suslin's conjecture is true for $\mathcal{Q}G$, i.e. $SK_1(\mathcal{Q}G) = 1$, then the uniqueness assertion of the main conjecture of equivariant Iwasawa theory is true.*

Proof: This is obvious by $SK_1(\mathcal{Q}G) = \ker(\text{Det})$. Thus, if $L_{K/k}$ has a preimage under Det , then this preimage is unique. \square

In Chapter 4, we will discuss Suslin's conjecture for $\mathcal{Q}G$ for profinite Galois groups $G = H \rtimes \Gamma$; more precisely, we reduce the conjecture to the case of $\mathcal{Q}^N U$ for pro- l groups $U = H \rtimes \Gamma$ and unramified extensions N/\mathbb{Q}_l . At this point, we list the cases for $\mathcal{Q}G$ which are known to have trivial reduced Whitehead group:

- (i) Let $G = H \times \Gamma$. Then $\mathcal{Q}G = (\mathcal{Q}\Gamma)[H] = \mathcal{Q}\Gamma \otimes_{\mathbb{Q}_l} \mathbb{Q}_l[H]$. By [45], we see $SK_1(\mathbb{Q}_l[H]) = 1$ because \mathbb{Q}_l is a local field. Furthermore, by [26], we conclude $SK_1(\mathcal{Q}G) = SK_1(\mathcal{Q}\Gamma \otimes_{\mathbb{Q}_l} \mathbb{Q}_l[H]) \cong SK_1(\mathbb{Q}_l[H]) = 1$ because $\mathcal{Q}\Gamma$ is a purely transcendental field extension of \mathbb{Q}_l .
- (ii) $SK_1(\mathcal{Q}G) = 1$ is true for a pro- l group G with abelian subgroup of index l by [36, p. 118].
- (iii) For $G = H \rtimes \Gamma$, where H is a finite group with $l \nmid |H|$, Suslin's conjecture is proven in [33, p. 169].

2 Reduction I: The main conjecture

The aim of this chapter is to reduce the proof of (MC) for arbitrary profinite Galois groups G to the case of a pro- l group G extended by finite unramified extensions of \mathbb{Q}_l . This is meant to justify the comment in [37] that the restriction in that paper to pro- l groups would not be necessary.

Note first that it suffices to consider G to be pro- l elementary, i.e. $G = \langle s \rangle \times U$ with a finite cyclic group $\langle s \rangle$ of order prime to l and a pro- l group U :

Proposition 2.1 (Ritter, Weiss) *Let G be profinite and assume that $\mu = 0$. The equivariant main conjecture, up to its uniqueness assertion, is true for G if and only if it is true for every pro- l elementary section $G(K'/k')$ of G with $k \subseteq k' \subseteq K' \subseteq K$ such that k'/k is finite and K/K' finite Galois.*

This is [33, Thm (C)].

For the rest of this chapter, G will denote a pro- l elementary group $G = \langle s \rangle \times U$ unless otherwise stated. Moreover, S will be assumed to be sufficiently large (in the sense of the previous chapter) and $\mu = 0$.

2.1 The logarithmic pseudomeasure

First we fix a finite set $\{\beta_i\}$ of representatives of the $G(\mathbb{Q}_l^c/\mathbb{Q}_l)$ -orbits of the \mathbb{Q}_l^c -irreducible characters of $\langle s \rangle$ and observe that the $N_i := \mathbb{Q}_l(\beta_i)$ are unramified over \mathbb{Q}_l as subextensions of the unramified extensions $\mathbb{Q}_l(\zeta_{|\langle s \rangle|})/\mathbb{Q}_l$. With $\mathfrak{o}_i := \mathbb{Z}_l[\beta_i]$ the ring of integers of N_i we thus have

$$\Lambda_\wedge G = (\mathbb{Z}_l[[G]])_\wedge = (\mathbb{Z}_l\langle s \rangle[[U]])_\wedge \cong \bigoplus_i (\mathbb{Z}_l[\beta_i][[U]])_\wedge = \bigoplus_i \Lambda_\wedge^{\mathfrak{o}_i} U$$

because the projective limit commutes with finite direct sums. Furthermore, we set

$$e_i := \frac{1}{|\langle s \rangle|} \left(\sum_{j \bmod |\langle s \rangle|} \mathrm{tr}_{N_i/\mathbb{Q}_l}(\beta_i(s^{-j})) s^j \right) \in \mathbb{Z}_l\langle s \rangle,$$

these are central idempotents of ΛG and

$$\bigoplus_i e_i \Lambda_\wedge G = \left(\bigoplus_i (e_i \mathbb{Z}_l \langle s \rangle) [[U]] \right)_\wedge \cong \left(\bigoplus_i \mathbb{Z}_l [\beta_i] [[U]] \right)_\wedge \cong \Lambda_\wedge G.$$

Since s and U commute, we have $\beta_i^u = \beta_i$ for all $u \in U$. Thus, we are in the situation of [33] with $U = U_i$ and $A_i = 1$. By Theorem 1 and its proof in [33], we get (by identifying Γ_k and Γ_{KU}) the following commutative diagram

$$\begin{array}{ccccc} K_1(\Lambda_\wedge G) & \longrightarrow & K_1(e_i \Lambda_\wedge G) & \xrightarrow[\cong]{\beta_i} & K_1(\Lambda_\wedge^{\mathfrak{o}_i} U) \\ \downarrow \text{Det} & & \downarrow \text{Det} & & \downarrow \text{Det} \\ \text{Hom}^*(R_l G, (\Lambda_\wedge^c \Gamma_k)^\times) & \longrightarrow & \text{Hom}^*(R_l^{(e_i)} G, (\Lambda_\wedge^c \Gamma_k)^\times) & \xrightarrow{\beta_i^*} & \text{Hom}^{N_i}(R_l U, (\Lambda_\wedge^c \Gamma_k)^\times) \end{array}$$

Here, $R_l^{(e_i)} G \subseteq R_l G$ denotes the \mathbb{Z} -span of the irreducible $\chi \in R_l G$ with $\chi(e_i) \neq 0$. In the left square, the horizontal maps are the usual restrictions to the direct summand $e_i \Lambda_\wedge G$, respectively to $R_l^{(e_i)} G$. In the right one, the top horizontal map is induced by the isomorphism $e_i \Lambda_\wedge G = (e_i \mathbb{Z}_l \langle s \rangle \otimes_{\mathbb{Z}_l} \Lambda U)_\wedge \cong \mathfrak{o}_i \otimes_{\mathbb{Z}_l} \Lambda_\wedge U = \Lambda_\wedge^{\mathfrak{o}_i} U$ and β_i^* is defined by

$$\beta_i^* : f \mapsto f_i, \quad f_i(\xi) = f(\beta_i \xi) \quad \forall \xi \in R_l U.$$

Lemma 2.1 *With the above notations, we have*

$$f \in \text{HOM}(R_l^{(e_i)} G, (\Lambda_\wedge^c \Gamma_k)^\times) \quad \Rightarrow \quad f_i = \beta_i^*(f) \in \text{HOM}^{N_i}(R_l U, (\Lambda_\wedge^c \Gamma_k)^\times).$$

Proof: We already know that $f_i = \beta_i^*(f) \in \text{Hom}^{N_i}(R_l U, (\Lambda_\wedge^c \Gamma_k)^\times)$ by the above commutative diagram. Thus, we only have to check that f_i satisfies the defining congruence of ‘HOM’. For this, we choose a $\xi \in R_l U$ and compute

$$f_i^{\text{Fr}}(\psi_l \xi) = f_i((\psi_l \xi)^{\varphi^{-1}})^\varphi = f(\beta_i(\psi_l \xi)^{\varphi^{-1}})^\varphi \stackrel{1}{=} f(\beta_i^\varphi(\psi_l \xi)) = f(\psi_l(\beta_i \xi))$$

where $\stackrel{1}{=}$ follows by the fact that f is equivariant under the action of $G(\mathbb{Q}_l^c/\mathbb{Q}_l)$ and thus under φ . Now, we see

$$\Psi(f_i^{\text{Fr}}(\psi_l \xi)) = \Psi(f(\psi_l(\beta_i \xi))) \equiv f(\beta_i \xi)^l \bmod (l \Lambda^c \Gamma_k) \equiv f_i(\xi)^l \bmod (l \Lambda^c \Gamma_k).$$

□

Analogously, the β_i induce the isomorphism

$$T(\mathcal{Q}_\wedge G) \cong \prod_i T(e_i \mathcal{Q}_\wedge G) \cong \prod_i T(\mathcal{Q}_\wedge^{N_i} U).$$

Now, we are ready to glue together the above diagram and diagram (1.1) with U replacing G . We get the commutative diagram

$$\begin{array}{ccccc}
K_1(\Lambda_\wedge G) & \xrightarrow[\cong]{\prod_i \beta_i} & \prod_i K_1(\Lambda_\wedge^{o_i} U) & \xrightarrow{\mathbf{L}} & \prod_i T(\mathcal{Q}_\wedge^{N_i} U) \\
\downarrow \text{Det} & & \downarrow \text{Det} & & \downarrow \cong \text{Tr} \\
\text{HOM}(R_l G, (\Lambda_\wedge^c \Gamma_k)^\times) & \xrightarrow{\prod_i \beta_i^*} & \prod_i \text{HOM}^{N_i}(R_l U, (\Lambda_\wedge^c \Gamma_k)^\times) & \xrightarrow{\mathbf{L}} & \prod_i \text{Hom}^{N_i}(R_l U, \mathcal{Q}_\wedge^c \Gamma_k)
\end{array} \tag{2.1}$$

with the maps of the right square defined componentwise. Because the L -function $L_{K/k}$ lies in $\text{HOM}(R_l G, (\Lambda_\wedge^c \Gamma_k)^\times)$, we obtain i -th components of $L_{K/k}$ and of the logarithmic pseudomeasure which we will see to have the same functorial properties as the objects from which they are derived:

Definition 2.1 • $L_i := (L_{K/k})_i := \beta_i^*(L_{K/k}) \in \text{HOM}^{N_i}(R_l U, (\Lambda_\wedge^c \Gamma_k)^\times)$ is called the i -th L -function.

• The unique element $t_i \in T(\mathcal{Q}_\wedge^{N_i} U)$ satisfying $\text{Tr}(t_i) = \mathbf{L}(L_i)$ is called the i -th logarithmic pseudomeasure.

L_i and t_i have the same functorial properties as $L_{K/k}$ and $t_{K/k}$:

Proposition 2.2 Set $U' \leq U$, $N \triangleleft U$ and $\tilde{U} := U/N$. Then

- $\text{res}_U^{U'} L_i = (L_{K/k'})_i =: L'_i$ and $\text{defl}_U^{\tilde{U}} L_i = (L_{\tilde{K}/k})_i =: \tilde{L}_i$,
- $\text{defl}_U^{\tilde{U}} t_i = (t_{\tilde{K}/k})_i =: \tilde{t}_i$.

Proof: We begin with the L -function. For $\tilde{G} := \langle s \rangle \times \tilde{U}$ and $\tilde{\chi} \in R_l \tilde{U}$ we compute

$$\begin{aligned}
(\text{defl}_U^{\tilde{U}} L_i)(\tilde{\chi}) &= L_i(\text{infl}_U^{\tilde{U}} \tilde{\chi}) = L_{K/k}(\beta_i \text{infl}_U^{\tilde{U}} \tilde{\chi}) = L_{K/k}(\text{infl}_G^{\tilde{G}}(\beta_i \tilde{\chi})) \\
&= (\text{defl}_G^{\tilde{G}} L_{K/k})(\beta_i \tilde{\chi}) = L_{\tilde{K}/k}(\beta_i \tilde{\chi}) = (L_{\tilde{K}/k})_i(\tilde{\chi}).
\end{aligned}$$

The computation for $\text{res}_U^{U'}$ being analogous, we go on with t_i . To show that t_i has the right behaviour under defl , we show that $\text{Tr}(\text{defl}_U^{\tilde{U}} t_i) = \mathbf{L}(\tilde{L}_i)$. The uniqueness of the logarithmic pseudomeasure then implies the claim. First, we need that Tr and $\text{defl}_U^{\tilde{U}}$ commute:

$$\begin{array}{ccc}
T(\mathcal{Q}_\wedge^{N_i} U) & \xrightarrow{\text{defl}_U^{\tilde{U}}} & T(\mathcal{Q}_\wedge^{N_i} \tilde{U}) \\
\downarrow \text{Tr} & & \downarrow \text{Tr} \\
\text{Hom}^{N_i}(R_l U, \mathcal{Q}_\wedge^c \Gamma_k) & \xrightarrow{\text{defl}_U^{\tilde{U}}} & \text{Hom}^{N_i}(R_l \tilde{U}, \mathcal{Q}_\wedge^c \Gamma_k)
\end{array}$$

For this, we adopt the proof of Lemma 2.1 in [37] to our situation. Choose a central open subset $\Gamma_0 \leq U$ and representatives u_i , $i = 1, \dots, s$, of the conjugacy classes of

U/Γ_0 . Every $x \in T(\mathcal{Q}_\wedge^{N_i}U)$ can be written as $x = \sum_{i=1}^s x_i \tau(u_i)$ with $x_i \in \mathcal{Q}_\wedge^{N_i}\Gamma_0$ and $\tau : \mathcal{Q}_\wedge^{N_i}U \rightarrow T(\mathcal{Q}_\wedge^{N_i}U)$ induced by the canonical projection; observe that the $\tau(u_i)$ form a $\mathcal{Q}_\wedge^{N_i}\Gamma_0$ -basis of $T(\mathcal{Q}_\wedge^{N_i}U)$. This implies $\text{defl}_{\tilde{U}}^{\tilde{U}}(x) = \sum_{i=1}^s \tilde{x}_i \tau(\tilde{u}_i)$ with \tilde{x}_i resp. \tilde{u}_i the images of x_i resp. u_i under the canonical projection $\mathcal{Q}_\wedge^{N_i}U \rightarrow \mathcal{Q}_\wedge^{N_i}\tilde{U}$.

Next, we choose an irreducible $\tilde{\chi} \in R_l(\tilde{U})$ and set $\chi := \text{infl}_{\tilde{U}}^U \tilde{\chi}$. In fact, we assume that $\tilde{\Gamma}_0 = \Gamma_0 \cap N$ lies in the kernel of this $\tilde{\chi}$ which is thus even a character of $\tilde{U}/\tilde{N} \cong U/\Gamma_0 \cdot N$. Then we show that the above diagram commutes for every such character and is stable under W-twist which finally shows that the diagram commutes for arbitrary characters. We are now ready for the computations:

$$\begin{aligned} (\text{Tr} \circ \text{defl}_{\tilde{U}}^{\tilde{U}})(x)(\tilde{\chi}) &= \text{Tr} \left(\sum_{i=1}^s \tilde{x}_i \tau(\tilde{u}_i) \right) (\tilde{\chi}) \\ &\stackrel{1}{=} \sum_{i=1}^s \overline{\tilde{x}_i} \overline{\tilde{u}_i} \tilde{\chi}(\tilde{u}_i) \stackrel{2}{=} \sum_{i=1}^s \overline{x_i} \overline{u_i} \chi(u_i). \end{aligned}$$

The equality $\stackrel{1}{=}$ follows from [34, Prop. 3]. Here, $\bar{\cdot}$ denotes the image in Γ_k . The equality $\stackrel{2}{=}$ is due to the definition of the action of $\text{defl}_{\tilde{U}}^{\tilde{U}}$ on characters. On the other side, we have

$$\begin{aligned} (\text{defl}_{\tilde{U}}^{\tilde{U}} \circ \text{Tr})(x)(\tilde{\chi}) &= \text{defl}_{\tilde{U}}^{\tilde{U}}(\text{Tr}x)(\tilde{\chi}) \\ &= (\text{Tr}x)(\text{infl}_{\tilde{U}}^U \tilde{\chi}) = \text{Tr}(x)(\chi) = \sum_{i=1}^s \overline{x_i} \overline{u_i} \chi(u_i) \end{aligned}$$

and thus the above diagram commutes for the action on characters whose kernel contains $\tilde{\Gamma}_0$.

Now we check the stability under W-twist. For this, let $\tilde{\rho}$ be an abelian character of \tilde{U} of type W. Then we have

$$\begin{aligned} (\text{Tr} \circ \text{defl}_{\tilde{U}}^{\tilde{U}})(x)(\tilde{\chi} \otimes \tilde{\rho}) &= \tilde{\rho}^\# \left((\text{Tr} \circ \text{defl}_{\tilde{U}}^{\tilde{U}})(x)(\tilde{\chi}) \right) \\ &= \tilde{\rho}^\# \left((\text{defl}_{\tilde{U}}^{\tilde{U}} \circ \text{Tr})(x)(\tilde{\chi}) \right) \\ &= (\text{defl}_{\tilde{U}}^{\tilde{U}} \circ \text{Tr})(x)(\tilde{\chi} \otimes \tilde{\rho}) \end{aligned}$$

because $(\text{Tr} \circ \text{defl})(x)$ and $(\text{defl} \circ \text{Tr})(x) \in \text{Hom}^{N_i}(R_l U, \mathcal{Q}_\wedge^c \Gamma_k)$ are compatible with W-twisting.

Finally, we show that $\text{Tr}(\text{defl}_{\tilde{U}}^{\tilde{U}} t_i) = \mathbf{L}(\tilde{L}_i)$ and thus t_i has the claimed behaviour under deflation:

$$\text{Tr}(\text{defl}_{\tilde{U}}^{\tilde{U}} t_i) = \text{defl}_{\tilde{U}}^{\tilde{U}}(\text{Tr}(t_i)) = \text{defl}_{\tilde{U}}^{\tilde{U}}(\mathbf{L}(L_i)) \stackrel{1}{=} \mathbf{L}(\text{defl}_{\tilde{U}}^{\tilde{U}} L_i) = \mathbf{L}(\tilde{L}_i).$$

For $\stackrel{1}{=}$ we use that \mathbf{L} commutes with defl because ψ_l and infl commute. \square

If G happens to be abelian, then Serre's pseudomeasure $\lambda_{K/k}$ has been seen to be a preimage of $L_{K/k}$ under Det . Next, we check that in the case of a pro- l elementary abelian G , this $\lambda_{K/k}$ gives rise to the i -th pseudomeasures which will be seen to be preimages of the i -th L -functions:

Definition 2.2 *Let G be abelian pro- l elementary, i.e. $G = \langle s \rangle \times U$ with $U = H_1 \times \Gamma$ abelian and $\Gamma \cong \mathbb{Z}_l$. Let $\lambda_{K/k} \in (\Lambda_\bullet G)^\times$ denote the pseudomeasure of K/k . Then, we define the i -th pseudomeasure λ_i by*

$$\Lambda_\bullet G = (\mathbb{Z}_l \langle s \rangle [H_1][[\Gamma]])_\bullet \xrightarrow{\cong} \oplus \mathbb{Z}_l [\beta_i][H_1][[\Gamma]]_\bullet, \quad \lambda_{K/k} \mapsto \beta_i(\lambda_{K/k}) =: \lambda_i.$$

Lemma 2.2 *With the notation of Definition 2.2, we obtain*

$$\text{Det}(\lambda_i) = L_i.$$

Proof: We check the claimed equality for arbitrary $\xi \in R_l U$:

$$\text{Det}(\lambda_i)(\xi) = \text{Det}(\beta_i(\lambda_{K/k}))(\xi) \stackrel{1}{=} \beta_i^*(\text{Det}(\lambda_{K/k}))(\xi) = \beta_i^*(L_{K/k})(\xi) = L_i(\xi).$$

For $\stackrel{1}{=}$ we have used that Det and the homomorphisms induced by β_i commute as seen above. \square

As a last tool to adopt the proof for pro- l groups to pro- l elementary groups, we need the following congruence:

Proposition 2.3 *If G has an abelian subgroup of index l , then*

$$\frac{\text{ver}(\lambda_i^{\text{ab}})^{\text{Fr}}}{\lambda'_i} \equiv 1 \pmod{\mathcal{T}'_i}$$

with $\lambda_i^{\text{ab}} := (\lambda^{\text{ab}})_i$, $\lambda'_i := (\lambda')_i$ and $\mathcal{T}'_i := \left\{ \sum_{j=0}^{l-1} y_i^{\alpha^j} : y_i \in \Lambda_{\wedge}^{\circ_i} U' \right\}$.

Remark 2.1 Because $G = \langle s \rangle \times U$, we see that every subgroup of index l is normal: Let $G' \leq G$ be such a subgroup. By $(l, |\langle s \rangle|) = 1$, we have $G' = \langle s \rangle \times U'$ with $U' \leq U$ an abelian subgroup of index l . Because U is a pro- l group, U' is normal and therefore G' is normal in G again by $(l, |\langle s \rangle|) = 1$.

For the proof of Proposition 2.3, we begin with the next

Lemma 2.3 *If G has an abelian subgroup of index l , then*

$$(\text{ver} \lambda^{\text{ab}})_i = \text{ver}(\lambda_i^{\text{ab}})^{\text{Fr}}.$$

Proof of Lemma 2.3: Let $G' \leq G$ be an abelian subgroup of index l . Since $G = \langle s \rangle \times U$, we see $G' = \langle s \rangle \times U'$ with $U' \leq U$ an abelian subgroup of index l , and $G^{\text{ab}} = \langle s \rangle \times U^{\text{ab}}$. As above, we set $A := \langle a \rangle := G/G' = U/U'$. Recall that the transfer map $\text{ver} : \Lambda_{\bullet} G \rightarrow \Lambda_{\bullet} G'$ (resp. $\text{ver} : \Lambda_{\bullet}^{\mathfrak{o}_i} U \rightarrow \Lambda_{\bullet}^{\mathfrak{o}_i} U'$) is induced by the transfer $G \rightarrow G^{\text{ab}} \rightarrow G'$ (resp. $U \rightarrow U^{\text{ab}} \rightarrow U'$), thus $\text{ver}(s) = \prod_{j=0}^{l-1} s^{a^j} = s^l$ because s is central in G .

By $\Lambda_{\bullet} G^{\text{ab}} = (\mathbb{Z}_l \langle s \rangle \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[[U^{\text{ab}}]])_{\bullet} = \mathbb{Z}_l \langle s \rangle \otimes_{\mathbb{Z}_l} \Lambda_{\bullet} U^{\text{ab}}$ and $\lambda^{\text{ab}} \in \Lambda_{\bullet} G^{\text{ab}}$, we achieve

$$\lambda^{\text{ab}} = \sum_x \left(\sum_{j=0}^{|\langle s \rangle|} x_j s^j \otimes x \right) \quad \text{with } x_j \in \mathbb{Z}_l, x \in \Lambda_{\bullet} U^{\text{ab}}$$

and a finite sum over certain x . Now, we are ready to compute the left hand side of the claimed equation:

$$\begin{aligned} (\text{ver} \lambda^{\text{ab}})_i &= \beta_i(\text{ver} \lambda^{\text{ab}}) \\ &= \beta_i \left(\sum_x \left(\sum_{j=0}^{|\langle s \rangle|} x_j s^{jl} \otimes \text{ver}(x) \right) \right) \\ &\stackrel{1}{=} \left(\sum_x \left(\sum_{j=0}^{|\langle s \rangle|} x_j \beta_i(s)^{jl} \otimes \text{ver}(x) \right) \right) \\ &\stackrel{2}{=} \left(\sum_x \left(\sum_{j=0}^{|\langle s \rangle|} x_j (\beta_i(s)^j)^{\text{Fr}} \otimes \text{ver}(x) \right) \right). \end{aligned}$$

For $\stackrel{1}{=}$, we have used that β_i is abelian and thus a homomorphism and, for $\stackrel{2}{=}$, observe that $\beta_i(s)^j \in \mathfrak{o}_i$ is a root of unity and $\text{Fr} \in (N_i/\mathbb{Q}_l)$ acts on roots of unity by taking to the l -th power.

For the right hand side we see

$$\begin{aligned} \text{ver}(\lambda_i^{\text{ab}})^{\text{Fr}} &= \text{ver} \left(\beta_i \left(\sum_x \left(\sum_{j=0}^{|\langle s \rangle|} x_j s^j \otimes x \right) \right) \right)^{\text{Fr}} \\ &= \text{ver} \left(\sum_x \left(\sum_{j=0}^{|\langle s \rangle|} x_j \beta_i(s)^j \otimes x \right) \right)^{\text{Fr}} \\ &= \left(\sum_x \left(\sum_{j=0}^{|\langle s \rangle|} x_j \beta_i(s)^j \otimes \text{ver}(x) \right) \right)^{\text{Fr}} \\ &= \left(\sum_x \left(\sum_{j=0}^{|\langle s \rangle|} x_j (\beta_i(s)^j)^{\text{Fr}} \otimes \text{ver}(x) \right) \right) \end{aligned}$$

which concludes the proof of the lemma. \square

Furthermore we need

Lemma 2.4 *If G has an abelian subgroup G' of index l , then*

$$\frac{\text{ver}(\lambda^{\text{ab}})}{\lambda'} \equiv 1 \pmod{\mathcal{T}'} \Rightarrow \frac{\text{ver}(\lambda_i^{\text{ab}})^{\text{Fr}}}{\lambda'_i} \equiv 1 \pmod{\mathcal{T}'_i}.$$

Proof of Lemma 2.4: Because β_i induces an isomorphism

$$\beta_i : \Lambda_{\wedge} G' e_i \xrightarrow{\cong} \Lambda_{\wedge}^{\mathfrak{o}_i} U',$$

it follows

$$\frac{\text{ver}(\lambda^{\text{ab}})}{\lambda'} \equiv 1 \pmod{\mathcal{T}'} \Rightarrow \frac{\beta_i(\text{ver}(\lambda^{\text{ab}}))}{\beta_i(\lambda')} \equiv 1 \pmod{\beta_i(\mathcal{T}')}.$$

In Lemma 2.3, we have shown that $\beta_i(\text{ver}(\lambda^{\text{ab}})) = (\text{ver} \lambda^{\text{ab}})_i = \text{ver}(\lambda_i^{\text{ab}})^{\text{Fr}}$, and moreover $\beta_i(\lambda') = \lambda'_i$ by definition. Thus, we only have to show that $\beta_i(\mathcal{T}') = \mathcal{T}'_i$. With $G/G' = U/U' =: \langle a \rangle$, this is seen easily:

$$\begin{aligned} \beta_i(\mathcal{T}') &= \beta_i \left(\left\{ \sum_{j=0}^{l-1} y^{a^j} : y \in \Lambda_{\wedge} G' \right\} \right) = \left\{ \beta_i \left(\sum_{j=0}^{l-1} y^{a^j} \right) : y \in \Lambda_{\wedge} G' \right\} \\ &= \left\{ \sum_{j=0}^{l-1} \beta_i(y)^{a^j} : y \in \Lambda_{\wedge} G' \right\} = \left\{ \sum_{j=0}^{l-1} y_i^{a^j} : y_i \in \Lambda_{\wedge}^{\mathfrak{o}_i} U' \right\} \\ &= \mathcal{T}'_i. \end{aligned}$$

□

Now we are ready for the

Proof of Proposition 2.3: Ritter and Weiss showed in [35] that

$$\text{ver}(\lambda^{\text{ab}}) \equiv \lambda' \pmod{\mathcal{T}'} \tag{2.2}$$

is true in the case that G has an abelian subgroup G' of index l . Then Lemma 2.4 implies the proposition. □

Finally we put together our work done so far:

Theorem 2.1 *Let G be a pro- l elementary group. Assume that S is sufficiently large and that $\mu = 0$. Then the equivariant main conjecture, up to uniqueness of Θ , is true if and only if every i -th logarithmic pseudomeasure t_i is integral, i.e. $t_i \in T(\Lambda^{\mathfrak{o}_i} U)$.*

Proof: The equivariant main conjecture is, up to its uniqueness assertion, equivalent to $L_{K/k} \in \text{Det}(K_1(\Lambda_{\wedge} G))$. This in turn is equivalent to $L_i \in \text{Det}(K_1(\Lambda_{\wedge}^{\mathfrak{o}_i} U))$ for all i by diagram (2.1). Thus, we have to show that $L_i \in \text{Det}(K_1(\Lambda_{\wedge}^{\mathfrak{o}_i} U))$ iff $t_i \in T(\Lambda^{\mathfrak{o}_i} U)$.

The proof of [37, Thm] now directly applies to our situation as we have seen that the used arguments are not affected by the extension to unramified extensions of \mathbb{Q}_l .

Namely, it uses induction on the index of G over its centre and exploits the functorial properties of the objects involved, i.e. the L -function, the pseudomeasure for abelian subgroups and factor groups and the logarithmic pseudomeasure. Via reduction and deflation these are put into relation and the congruence (2.2) finishes the proof, provided (MC) holds, up to uniqueness, whenever G has an abelian subgroup of index l . Thus, it remains to check whether this remains true for pro- l elementary groups with abelian subgroup of index l . This will be done in the next section. (\square)

The philosophy of this section is that whenever the proof of (MC) for a pro- l group U does not employ properties of \mathbb{Z}_l which do not extend to unramified extensions $\mathbb{Z}_l[\beta]$, (MC) holds for all pro- l elementary groups with Sylow l -subgroup isomorphic to U . If the class \mathfrak{U} of those groups U is closed under taking open subgroups and factor groups with respect to finite normal subgroups, then (MC) is true for G as long as the Sylow l -subgroups of G belong to \mathfrak{U} .

As an example, we show that the equivariant main conjecture, up to uniqueness of Θ , holds whenever the l -part U of G has an abelian subgroup of index l . This is our aim in the next section.

2.2 Groups with abelian subgroups of index l

The proof of (MC) for pro- l groups with abelian subgroups of index l is the subject of [36]. This work can be generalized to

Theorem 2.2 *Let $G = \langle s \rangle \times U$ be a pro- l elementary group. Assume that S is sufficiently large and that $\mu = 0$. Assume further that U has an abelian subgroup of index l . Then, every i -th logarithmic pseudomeasure t_i is integral, i.e. $t_i \in \mathbb{T}(\Lambda^{\circ i} U)$. Therefore, the equivariant main conjecture, up to uniqueness of Θ , is true for G .*

First, we show that the map Res can be generalized to the pro- l elementary situation.

Definition 2.3 *Let U be a pro- l group. Assume that U has an abelian subgroup U' of index l . We define the restriction map*

$$\begin{aligned} \text{Res}_U^{U'} : \text{Hom}^{N_i}(R_l U, \mathcal{Q}_{\wedge}^c \Gamma_k) &\rightarrow \text{Hom}^{N_i}(R_l U', \mathcal{Q}_{\wedge}^c \Gamma_{k'}), \\ f &\mapsto [\chi' \mapsto f(\text{ind}_U^{U'} \chi') + \sum_{r \geq 1} \frac{\Psi^r}{l^r} (f^{\text{Fr}^r}(\psi_l^{r-1} \chi))], \end{aligned}$$

with $\chi = \psi_l(\text{ind}_{U'}^U \chi') - \text{ind}_{U'}^U(\psi_l \chi')$.

Lemma 2.5 *Let U be a pro- l group. Assume that U has an abelian subgroup U' of index l . Then, the diagram*

$$\begin{array}{ccccc} K_1(\Lambda_{\wedge}^{\mathfrak{o}_i} U) & \xrightarrow{\mathbb{L}} & T(\mathcal{Q}_{\wedge}^{N_i} U) & \xrightarrow{\text{Tr}} & \text{Hom}^{N_i}(R_l U, \mathcal{Q}_{\wedge}^c \Gamma_k) \\ \downarrow \text{res}_U^{U'} & & \downarrow \text{Res}_U^{U'} & & \downarrow \text{Res}_U^{U'} \\ K_1(\Lambda_{\wedge}^{\mathfrak{o}_i} U') & \xrightarrow{\mathbb{L}'} & T(\mathcal{Q}_{\wedge}^{N_i} U') & \xrightarrow{\text{Tr}'} & \text{Hom}^{N_i}(R_l U', \mathcal{Q}_{\wedge}^c \Gamma_{k'}) \end{array} \quad (2.3)$$

commutes. Here, \mathbb{L}' resp. Tr' stand for the logarithm resp. the trace map on the U' -level.

Moreover, the following diagram commutes:

$$\begin{array}{ccc} \text{HOM}^{N_i}(R_l U, (\Lambda_{\wedge}^c \Gamma_k)^{\times}) & \xrightarrow{\mathbf{L}} & \text{Hom}^{N_i}(R_l U, \mathcal{Q}_{\wedge}^c \Gamma_k) \\ \downarrow \text{res}_U^{U'} & & \downarrow \text{Res}_U^{U'} \\ \text{HOM}^{N_i}(R_l U', (\Lambda_{\wedge}^c \Gamma_{k'})^{\times}) & \xrightarrow{\mathbf{L}'} & \text{Hom}^{N_i}(R_l U', \mathcal{Q}_{\wedge}^c \Gamma_{k'}) \end{array} \quad (2.4)$$

with again \mathbf{L}' the logarithm on the U' -level.

Observe that the Galois group $G(\mathbb{Q}_l^c/N_i)$ depends only on N_i , i.e. on $\langle s \rangle$ and thus remains unaffected by the restriction to $G' = \langle s \rangle \times U'$. Moreover, the restriction map Res on the $T(\cdot)$ -level is defined by diagram (2.3) via the trace isomorphism. Thus the right square of the upper diagram commutes by definition.

Proof: We first have to check that $\text{Res}_U^{U'} f$ lies in $\text{Hom}^{N_i}(R_l U', \mathcal{Q}_{\wedge}^c \Gamma_{k'})$ for all homomorphisms $f \in \text{Hom}^{N_i}(R_l U, \mathcal{Q}_{\wedge}^c \Gamma_k)$. For the Galois equivariance, we compute for a $\sigma \in G(\mathbb{Q}_l^c/N_i)$ that

$$\begin{aligned} (\text{Res}_U^{U'} f)(\chi)^{\sigma} &= \left(f(\text{ind}_U^{U'} \chi') + \sum_{r \geq 1} \frac{\Psi^r}{l^r} (f^{\text{Fr}^r}(\psi_l^{r-1} \chi)) \right)^{\sigma} \\ &= f(\text{ind}_U^{U'} \chi')^{\sigma} + \sum_{r \geq 1} \left(\frac{\Psi^r}{l^r} (f^{\text{Fr}^r}(\psi_l^{r-1} \chi)) \right)^{\sigma} \\ &= (\text{res}_U^U f)(\chi')^{\sigma} + \sum_{r \geq 1} \frac{1}{l^r} \left((\Psi^r f^{\text{Fr}^r})(\psi_l^{r-1} \chi) \right)^{\sigma} \\ &\stackrel{1}{=} (\text{res}_U^U f)(\chi'^{\sigma}) + \sum_{r \geq 1} \frac{1}{l^r} (\Psi^r f^{\text{Fr}^r})(\psi_l^{r-1} \chi^{\sigma}) \\ &= (\text{Res}_U^{U'} f)(\chi^{\sigma}). \end{aligned}$$

For $\stackrel{1}{=}$, we use that $\Psi^r f^{\text{Fr}^r} \in \text{Hom}^{N_i}(R_l U, \mathcal{Q}_{\wedge}^c \Gamma_k)$ because $\Psi : \Lambda^{\mathfrak{o}_i} \Gamma_k \rightarrow \Lambda^{\mathfrak{o}_i} \Gamma_k$ is a homomorphism stable under the Galois action as Ψ only acts on the group elements but not on the coefficients. Moreover, the Adams operation is stable under the Galois action, too.

Next, we refer to [36], where the compatibility with W -twist is shown as well as the fact that the restricted f takes its values in $\mathcal{Q}_\Lambda^c \Gamma_k$. This can be adapted to our situation easily.

Now, we consider the commutativity of the latter diagram: For $\chi' \in R_l U'$ and $f \in \text{HOM}^{N_i}(R_l U, (\Lambda_\Lambda^c \Gamma_k)^\times)$, we compute

$$\begin{aligned}
(\text{Res}_U^{U'} \mathbf{L}f)(\chi') &= (\mathbf{L}f)(\text{ind}_{U'}^U \chi') + \sum_{r \geq 1} \frac{\Psi^r}{l^r} [(\mathbf{L}f)^{\text{Fr}^r}(\psi_l^{r-1} \chi)] \\
&= (\mathbf{L}f)(\text{ind}_{U'}^U \chi') + \sum_{r \geq 1} \frac{\Psi^r}{l^r} [(\mathbf{L}f^{\text{Fr}^r})(\psi_l^{r-1} \chi)] \\
&= (\mathbf{L}f)(\text{ind}_{U'}^U \chi') + \sum_{r \geq 1} \frac{\Psi^r}{l^r} \left[\frac{1}{l} \log \frac{f^{\text{Fr}^r}(\psi_l^{r-1} \chi)^l}{\Psi(f^{\text{Fr}^{r+1}}(\psi_l^r \chi))} \right] \\
&\stackrel{1}{=} (\mathbf{L}f)(\text{ind}_{U'}^U \chi') + \sum_{r \geq 1} \frac{\Psi^r}{l^r} [\log(f^{\text{Fr}^r}(\psi_l^{r-1} \chi)) - \frac{\Psi}{l} \log(f^{\text{Fr}^{r+1}}(\psi_l^r \chi))] \\
&= (\mathbf{L}f)(\text{ind}_{U'}^U \chi') + \sum_{r \geq 1} \frac{\Psi^r}{l^r} \log(f^{\text{Fr}^r}(\psi_l^{r-1} \chi)) - \sum_{r \geq 2} \frac{\Psi^r}{l^r} \log(f^{\text{Fr}^r}(\psi_l^{r-1} \chi)) \\
&= (\mathbf{L}f)(\text{ind}_{U'}^U \chi') + \frac{\Psi}{l} \log(f^{\text{Fr}}(\chi)) \\
&= \frac{1}{l} \log \frac{f(\text{ind}_{U'}^U \chi')^l}{\Psi(f^{\text{Fr}}(\psi_l \text{ind}_{U'}^U \chi'))} + \frac{\Psi}{l} \log \frac{f^{\text{Fr}}(\psi_l \text{ind}_{U'}^U \chi')}{f^{\text{Fr}}(\text{ind}_{U'}^U \psi_l \chi')} \\
&= \frac{1}{l} \log \frac{(f(\text{ind}_{U'}^U \chi')^l) \cdot \Psi(f^{\text{Fr}}(\psi_l \text{ind}_{U'}^U \chi'))}{\Psi(f^{\text{Fr}}(\psi_l \text{ind}_{U'}^U \chi')) \cdot \Psi(f^{\text{Fr}}(\text{ind}_{U'}^U \psi_l \chi'))} \\
&= \frac{1}{l} \log \frac{f(\text{ind}_{U'}^U \chi')^l}{\Psi(f^{\text{Fr}}(\text{ind}_{U'}^U \psi_l \chi'))} \\
&= \frac{1}{l} \log \frac{(\text{res}_U^{U'} f)(\chi')^l}{\Psi((\text{res}_U^{U'} f^{\text{Fr}})(\psi_l \chi'))} \\
&= (\mathbf{L}' \text{res}_U^{U'} f)(\chi').
\end{aligned}$$

In fact, this is exactly the adaptation of the computation in the pro- l situation. We have copied it in order to explain why the Frobenius automorphism occurs with the power r in the modified restriction map Res . Here, we are allowed to split the logarithm in $\stackrel{1}{=}$ because $\frac{f(\chi)^l}{\Psi(f^{\text{Fr}}(\psi_l \chi))} \equiv 1 \pmod{l \Lambda_\Lambda^c \Gamma_k}$ for all $\chi \in R_l U$ and $\chi(1) = 0$ by construction:

$$f(\psi_l^{r-1} \chi)^{l^s} \equiv \Psi^s f^{\text{Fr}^s}(\psi_l^s \psi_l^{r-1} \chi) \equiv \Psi^s f^{\text{Fr}^s}(\chi(1)) = 1 \pmod{l \Lambda_\Lambda^c \Gamma_k}$$

for $s \in \mathbb{N}$ big enough.

Finally, we can easily show the commutativity of the left square of the upper diagram. We glue together the diagrams (1.1), (2.3) and (2.4) and use the fact that Det commutes with res (see [34]). \square

Next, we see that the logarithmic pseudomeasure behaves in the right way under Res :

Lemma 2.6 *Let $G = \langle s \rangle \times U = G(K/k)$. Let k' be the subfield of K fixed by U' . Then*

$$\text{Res}_U^{U'} t_i = (t_{K/k'})_i =: (t')_i.$$

Proof: This follows from the commutative diagram (2.3) and the result $\text{res}_U^{U'} L_i = L'_i$ of Proposition 2.2. \square

From now on, we set $A = G/G' = U/U' = \langle a \rangle$ and observe that a acts on G' and U' via conjugation.

Lemma 2.7 *Let $\tau : \Lambda_\wedge^{\circ_i} U \rightarrow T(\Lambda_\wedge^{\circ_i} U)$ denote the canonical map and $u \in U$. If U' is an abelian subgroup of index l in U , then*

$$\text{Res}_U^{U'} (\tau(u)) = \begin{cases} \sum_{i=0}^{l-1} u^{a^i} & \text{if } u \in U', \\ u^l & \text{else.} \end{cases}$$

Proof: This is obviously true because it is true for $G = U$ pro- l ([36]) and only group elements are regarded. \square

An easy generalization can be stated: If $\Gamma_0 \cong \mathbb{Z}_l$ is a central subgroup of U contained in U' , then the elements of $T(\Lambda_\wedge^{\circ_i} U)$ can uniquely be written as $\sum_u \beta_u \tau(u)$ with $\beta_u \in \Lambda_\wedge \Gamma_0$ and u running through a set of preimages of the conjugacy classes of U/Γ_0 . For each summand, we have

$$\text{Res}_U^{U'} (\beta_u \tau(u)) = \begin{cases} \sum_{i=0}^{l-1} \beta_u u^{a^i} & \text{if } u \in U', \\ \Psi(\beta_u^{\text{Fr}}) u^l & \text{else.} \end{cases}$$

Now we possess all instruments to prove the main theorem. We outline the steps of this proof which imitates the proof of the analogous theorem in the pro- l situation stated in [36].

Step 1 *Assume that there exists an $x_i \in T(\Lambda_\wedge^{\circ_i} U)$ such that $\text{defl}_U^{U^{\text{ab}}} x_i = \text{defl}_U^{U^{\text{ab}}} t_i$ and $\text{Res}_U^{U'} x_i = \text{Res}_U^{U'} t_i$. Then $t_i \in T(\Lambda_\wedge^{\circ_i} U)$.*

The proof that this is true can be copied word for word from the pro- l situation.

Step 2 *There exists an $x'_i \in T(\Lambda_\wedge^{\circ_i} U)$ with $\text{defl}_U^{U^{\text{ab}}} x'_i = \text{defl}_U^{U^{\text{ab}}} t_i$.*

To see this, we first introduce some notation. Let \mathfrak{a}_\wedge denote the kernel of the deflation map $\text{defl}_U^{U^{\text{ab}}} : \Lambda_\wedge^{\circ_i} U \rightarrow \Lambda_\wedge^{\circ_i} U^{\text{ab}}$ and thus $\tau(\mathfrak{a}_\wedge)$ is the kernel of the deflation map $\text{defl}_U^{U^{\text{ab}}} : T(\Lambda_\wedge^{\circ_i} U) \rightarrow T(\Lambda_\wedge^{\circ_i} U^{\text{ab}}) = \Lambda_\wedge^{\circ_i} U^{\text{ab}}$. Furthermore we define \mathfrak{b}'_\wedge to be the

kernel of the projection $\Lambda_{\wedge}^{\circ_i} U' \twoheadrightarrow \Lambda_{\wedge}^{\circ_i} U' / [U, U]$. The claim of the second step now follows from the following diagram:

$$\begin{array}{ccccc}
 \tau(\mathfrak{a}_{\wedge})^{\subset} & \longrightarrow & T(\Lambda_{\wedge}^{\circ_i} U) & \twoheadrightarrow & \Lambda_{\wedge}^{\circ_i} U^{\text{ab}} \\
 \downarrow \text{Res}_{U'}^{U'} & & \downarrow \text{Res}_{U'}^{U'} & & \downarrow \text{Res}_{U'}^{U'} \\
 (\mathfrak{b}'_{\wedge})^A & \longrightarrow & (\Lambda_{\wedge}^{\circ_i} U')^A & \longrightarrow & (\Lambda_{\wedge}^{\circ_i} (U' / [U, U]))^A \\
 \downarrow & & \downarrow & & \downarrow \\
 \hat{H}^0(A, \mathfrak{b}'_{\wedge})^{\subset} & \longrightarrow & \hat{H}^0(A, \Lambda_{\wedge}^{\circ_i} U') & \longrightarrow & \hat{H}^0(A, (\Lambda_{\wedge}^{\circ_i} (U' / [U, U])))
 \end{array} \tag{2.5}$$

The lower vertical maps are defined in the canonical way. Here, the computations that the rows and the left vertical column are exact and that the left bottom horizontal map is injective can be translated from the pro- l situation.

Step 3 With x'_i as above, we set $\text{Res}_U^{U^{\text{ab}}} x'_i = t_i + x''_i$. It is possible to change x'_i modulo the kernel of $\text{defl}_U^{U^{\text{ab}}}$ such that the new x''_i becomes zero iff $x'_i \in \mathcal{T}'$.

As in the pro- l situation, we can show that this x''_i is fixed under the action of A . The claim thus follows by diagram (2.5).

Step 4 $x'_i \in \mathcal{T}'$ is achievable.

To find this x'_i , we consider the situation on the multiplicative (i.e. left) side of diagram (1.1). Because U' and U^{ab} are abelian, there exist pseudomeasures λ'_i and λ_i^{ab} satisfying $\text{Det}(\lambda'_i) = L'_i$ and $\text{Det}(\lambda_i^{\text{ab}}) = L_i^{\text{ab}}$. Furthermore, we know that

$$\frac{\text{ver}(\lambda_i^{\text{ab}})^{\text{Fr}}}{\lambda'_i} \equiv 1 \pmod{\mathcal{T}'_i}.$$

Let $y_i \in (\Lambda_{\wedge}^{\circ_i} U)^{\times}$ have $\text{defl}_U^{U^{\text{ab}}} y_i = \lambda_i^{\text{ab}}$ and set $\text{res}_U^{U'} y_i = \lambda'_i \cdot y'_i$. Then

$$y'_i = \frac{\text{res}_U^{U'} y_i}{\lambda'_i} \equiv \frac{\text{ver}(\lambda_i^{\text{ab}})^{\text{Fr}}}{\lambda'_i} \equiv 1 \pmod{\mathcal{T}'_i}$$

as seen in the proof of [37, Prop. 3.2] with the generalization to unramified extensions N_i/\mathbb{Q}_l due to [34, p. 47]. Analogously to the proof in [36], we moreover achieve

$$y'_i \in 1 + \text{tr}_A \mathfrak{b}'_{\wedge}.$$

To go back to the additive (i.e. right) side of diagram (1.1), we apply the logarithm map and see

$$\mathbb{L}'(y'_i) = \frac{1}{l} \log \frac{y_i^l}{\Psi(y_i^{\text{Fr}})}$$

by [34, p. 39] and thus $\mathbb{L}'(y'_i) \in \mathcal{T}'$ if

$$\frac{y_i^l}{\Psi(y_i^{\text{Fr}})} \equiv 1 \pmod{l\mathcal{T}'}$$

We write $y'_i = 1 + \text{tr}_A \beta'_i$ with $\beta'_i \in \mathfrak{b}'_\Lambda$. Then this last congruence is equivalent to

$$(\text{tr}_A \beta'_i)^l \equiv \Psi((\text{tr}_A \beta'_i)^{\text{Fr}}) \pmod{l\mathcal{T}'} \quad (2.6)$$

because $(1 + \text{tr}_A \beta'_i)^l \equiv 1 + (\text{tr}_A \beta'_i)^l \pmod{l\mathcal{T}'}$ and Ψ is a homomorphism for the abelian group U' .

Next, we pick a central subgroup $\Gamma_0 \cong \mathbb{Z}_l$ of U and write $\beta'_i = \sum_{u',c} \beta_{u',c} u'(c-1)$ with elements $\beta_{u',c} \in \Lambda^{\mathfrak{o}_i} \Gamma_0$, $u' \in U'$ and $c \in [U, U]$ (observe that these $u'(c-1)$ generate \mathfrak{b}'_Λ). Thus, the left hand side of (2.6) is

$$\begin{aligned} (\text{tr}_A \beta'_i)^l &= \left(\sum_{u',c} \beta_{u',c} \text{tr}_A(u'(c-1)) \right)^l \equiv \sum_{u',c} (\beta_{u',c})^l (\text{tr}_A(u'(c-1)))^l \\ &\equiv \sum_{u',c} \Psi(\beta_{u',c}^{\text{Fr}}) \left((\text{tr}_A(u'c))^l - (\text{tr}_A u')^l \right) \pmod{l\mathcal{T}'} \end{aligned}$$

For the right hand side of (2.6), we find

$$\Psi((\text{tr}_A \beta'_i)^{\text{Fr}}) = \sum_{u',c} \Psi(\beta_{u',c}^{\text{Fr}}) \left((\text{tr}_A(u'c))^l - (\text{tr}_A u')^l \right)$$

because Ψ and tr_A commute. For the rest of the computations, we are allowed to copy the computations of [36] literally. Thus, the theorem is proven. \square

2.3 The choice of the set of places S

In [37], it is shown for pro- l groups G that (MC) does not depend on the choice of the set of places S , as long as S is large enough. Of course a modification of S changes the L -function and \mathfrak{U} , and thus the linking element Θ does depend on S . But the question whether there exists such a Θ can be answered simultaneously for any large enough set S .

In fact, the only argument which needs the pro- l assumption is that for every irreducible character $\chi \in R_l G$, there exists an open subgroup G' and an abelian irreducible character $\chi' \in R_l G'$ such that $\chi = \text{ind}_{G'}^G \chi'$. This applies to our pro- l elementary group G as well because G is nilpotent. Thus, we can state without further work

Theorem 2.3 *The equivariant main conjecture does not depend on the choice of a sufficiently large set of places S .* \square

3 The structure of $\mathcal{Q}G$

In this chapter, we restrict ourselves to pro- l Galois groups G and compute the structure of $\mathcal{Q}G$ for these groups.

Recall that $G = H \rtimes \Gamma$ with $\Gamma = \langle \gamma \rangle \cong G(k_\infty/k)$ and H a finite l -group. There exists an $m \in \mathbb{N}$ such that $\Gamma_0 = \Gamma^{l^m}$ is a central subgroup of G and then

$$\mathcal{Q}G = \bigoplus_{i=0}^{l^m-1} (\mathcal{Q}\Gamma_0)[H]\gamma^i.$$

Furthermore, with η an absolutely irreducible constituent of $\text{res}_G^H(\chi)$, we recollect the definitions

$$\begin{aligned} St(\eta) &= \{g \in G : \eta^g = \eta\}, \quad w_\chi = [G : St(\eta)], \\ e(\eta) &= \frac{\eta(1)}{|H|} \sum_{h \in H} \eta(h^{-1})h \quad \text{and} \quad e_\chi = \sum_{\eta | \text{res}_G^H \chi} e(\eta). \end{aligned}$$

We have already seen that $\mathcal{Q}G$ is a finite dimensional semisimple $\mathcal{Q}\Gamma_0$ -algebra. Let A be a simple component of $\mathcal{Q}G$. Then, A corresponds to an irreducible $\chi \in R_l G$ (modulo W -twists and the action of the Galois group $G(\mathbb{Q}_l^c/\mathbb{Q}_l)$) by $Ae_\chi \neq 0$. From now on, we fix a representative χ in the orbit under W -twisting.

For the investigation of the structure of A , we need the following property:

Lemma 3.1 *Let A be a simple component of $\mathcal{Q}G = \bigoplus_{i=0}^{l^m-1} (\mathcal{Q}\Gamma_0)[H]\gamma^i$. Then*

$$A \cap (\mathcal{Q}\Gamma_0)[H] \neq 0.$$

Proof: First, we know that there exists a primitive central idempotent $e \in \mathcal{Q}G$ with $A = \mathcal{Q}Ge$. Furthermore, the tensor product $\mathbb{Q}_l^c \otimes_{\mathbb{Q}_l} A = \mathbb{Q}_l^c \otimes_{\mathbb{Q}_l} \mathcal{Q}Ge$ is the direct sum of some simple components which correspond to central primitive idempotents $e_{\chi_\nu} \in \mathcal{Q}^c G$ for a finite set of irreducible $\chi_\nu \in R_l G$. Thus, we get

$$\mathbb{Q}_l^c \otimes_{\mathbb{Q}_l} A = \mathbb{Q}_l^c \otimes_{\mathbb{Q}_l} \mathcal{Q}Ge = \bigoplus_{\nu} \mathcal{Q}^c G e_{\chi_\nu}.$$

As the e_{χ_ν} are exactly the primitive central idempotents lying over e , we conclude $e = \sum_{\nu} e_{\chi_\nu} \in A$. Therefore, by

$$e_{\chi_\nu} = \sum_{\eta | \text{res}_G^H \chi_\nu} e(\eta) = \sum_{\eta | \text{res}_G^H \chi_\nu} \left(\frac{\eta(1)}{|H|} \sum_{h \in H} \eta(h^{-1})h \right) \in (\mathcal{Q}^c \Gamma_0)[H],$$

we achieve

$$e = \sum_{\nu} e_{\chi_{\nu}} \in A \cap (\mathcal{Q}\Gamma_0)[H].$$

Since $e \in A \subseteq \mathcal{Q}G$ is invariant under the action of $G(\mathbb{Q}_l^c/\mathbb{Q}_l)$, we get $e \in (\mathcal{Q}\Gamma_0)[H]$ and finally

$$e \in A \cap (\mathcal{Q}\Gamma_0)[H].$$

This proves the lemma. \square

Now, $B := A \cap (\mathcal{Q}\Gamma_0)[H]$ is a two-sided ideal of $(\mathcal{Q}\Gamma_0)[H]$ (because A is a two-sided ideal in $\mathcal{Q}G$) and is thus the sum of some Wedderburn components of $(\mathcal{Q}\Gamma_0)[H]$. Because of $Ae_{\chi} \neq 0$ and

$$(\mathcal{Q}\Gamma_0)[H]e_{\chi} = (\mathcal{Q}\Gamma_0)[H] \sum_{\eta | \text{res}_G^H \chi} e(\eta) \neq 0,$$

there is a Wedderburn component B_0 of $(\mathcal{Q}\Gamma_0)[H]$ in B with $B_0e_{\chi} \neq 0$. Clifford theory ([13, p. 565]) shows that

$$\text{res}_G^H(\chi) = \sum_{j=0}^{w_{\chi}-1} \eta^{\gamma^j},$$

where γ^j runs through a set of representatives of $G/St(\eta)$. Thus, without loss of generality, we can assume $B_0e(\eta) \neq 0$ because $e_{\chi} = \sum_{j=0}^{w_{\chi}-1} e(\eta^{\gamma^j})$.

But $e(\eta) = \frac{\eta(1)}{|H|} \sum_{h \in H} \eta(h^{-1})h \notin (\mathcal{Q}\Gamma_0)[H]$ because $\eta(H) \not\subseteq \mathbb{Q}_l$. With

$$\text{Gal} := G(\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0/\mathcal{Q}\Gamma_0),$$

we therefore have

$$B_0 = (\mathcal{Q}\Gamma_0)[H] \sum_{\sigma \in \text{Gal}} e(\eta^{\sigma}) =: (\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta)$$

and $\varepsilon(\eta)$ is the central primitive idempotent of B_0 . Observe that, by the definition of $\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0$, the Galois group $\text{Gal} = G(\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0/\mathcal{Q}\Gamma_0)$ is canonically isomorphic to $G(\mathbb{Q}_l(\eta)/\mathbb{Q}_l)$. Thus, Gal is cyclic because $l \neq 2$.

The other Wedderburn components of B belong to the other irreducible characters η^{γ^j} because these are exactly the characters not annihilating B . Thus, the structure of B is given by

$$B = \bigoplus_{j=0}^{v_{\chi}-1} (\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta^{\gamma^j})$$

with

$$v_{\chi} = \min\{0 \leq j \leq w_{\chi} - 1 : \eta^{\gamma^j} = \eta^{\sigma} \text{ for some } \sigma \in \text{Gal}\};$$

note that $v_\chi \mid w_\chi$. We need v_χ to avoid that some $e(\eta^{\gamma^j})$ appears more than once as summand of the central primitive idempotents of B . Thus, the choice of v_χ ensures that the sum is direct.

By Roquette ([39]), the direct summands of B have trivial Schur index and are thus full matrix rings over certain fields. More precisely, we achieve

$$B \cong \bigoplus_{j=0}^{v_\chi-1} (\mathcal{Q}^{\mathbb{Q}_l(\eta^{\gamma^j})} \Gamma_0)_{\eta^{\gamma^j}(1) \times \eta^{\gamma^j}(1)} = \bigoplus_{j=0}^{v_\chi-1} (\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma_0)_{\eta(1) \times \eta(1)}$$

and

$$\begin{aligned} A &\stackrel{1}{=} \bigoplus_{i=0}^{l^m-1} B\gamma^i = \bigoplus_{i=0}^{l^m-1} \left(\bigoplus_{j=0}^{v_\chi-1} (\mathcal{Q} \Gamma_0)[H]\varepsilon(\eta^{\gamma^j}) \right) \gamma^i \\ &\cong \bigoplus_{i=0}^{l^m-1} \left(\bigoplus_{j=0}^{v_\chi-1} (\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma_0)_{\eta(1) \times \eta(1)} \right) \gamma^i. \end{aligned}$$

Here, we have used for $\stackrel{1}{=}$ that, on the one hand, the direct sum is contained in the $\mathcal{Q}\Gamma_0$ -algebra A by the definition of B . On the other hand, this direct sum is, as a two-sided ideal of $\mathcal{Q}G$, the direct sum of some Wedderburn components. Because A is simple, the equality holds.

Next, we define

$$G_0 := \{\sigma \in \text{Gal} : \eta^\sigma = \eta^{\gamma^j} \text{ for a } 0 \leq j \leq w_\chi - 1\}.$$

This group is strongly related to v_χ : The minimal $0 \leq j \leq w_\chi - 1$ satisfying the condition $\eta^\sigma = \eta^{\gamma^j}$ for a $\sigma \in \text{Gal}$ is by definition v_χ .

Proposition 3.1 *Let A be the simple component of $\mathcal{Q}G$ corresponding to the irreducible character $\chi \in R_l G$. Then,*

$$A \cong \bigoplus_{i=0}^{l^m-1} \left(\bigoplus_{j=0}^{v_\chi-1} (\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma_0)_{\eta(1) \times \eta(1)} \right) \gamma^i$$

has centre

$$Z(A) \cong \mathcal{Q}^L \Gamma^{w_\chi}$$

with $L = \mathbb{Q}_l(\eta)^{G_0}$ and $G_0 = \{\sigma \in \text{Gal} : \eta^\sigma = \eta^{\gamma^j} \text{ for a } 0 \leq j \leq w_\chi - 1\}$, as before. Moreover, G_0 is a cyclic group of order $\frac{w_\chi}{v_\chi}$.

Proof: First, we examine G_0 . Define $\sigma_{v_\chi} \in \text{Gal}$ via $\eta^{\gamma^{v_\chi}} = \eta^{\sigma_{v_\chi}}$. Observe that the actions of γ and $\sigma \in \text{Gal}$ commute. Thus, induction yields $\eta^{\gamma^{nv_\chi}} = \eta^{\sigma_{v_\chi}^n}$ for $0 \leq n \leq w_\chi/v_\chi - 1$ and therefore, by $\eta^{\gamma^{v_\chi \cdot w_\chi/v_\chi}} = \eta^{\gamma^{w_\chi}} = \eta$, the order of σ_{v_χ} is w_χ/v_χ .

Furthermore, G_0 is, as subgroup of the cyclic group $\text{Gal} \cong G(\mathbb{Q}_l(\eta)/\mathbb{Q}_l)$, cyclic. Next, every $0 \leq j \leq v_\chi - 1$ such that there exists a $\sigma \in \text{Gal}$ with $\eta^\sigma = \eta^{\gamma^j}$ is a multiple of v_χ , otherwise v_χ would not be minimal. Thus, we conclude that σ_{v_χ} is the generator of G_0 , we obtain $G_0 = \langle \sigma_{v_\chi} \rangle$.

Next, we compute the centre of A . To do so, take a central element $z = \sum_{i=0}^{l^m-1} b_i \gamma^i$ with $b_i = \sum_{j=0}^{v_\chi-1} \beta_{ij} \varepsilon(\eta^{\gamma^j}) \in B$. Observe that the i -sum is not orthogonal and thus

$$\gamma z \stackrel{!}{=} z \gamma \iff \sum_{i=0}^{l^m-1} \gamma b_i \gamma^i = \sum_{i=0}^{l^m-1} b_i^{\gamma^{-1}} \gamma^{i+1} \stackrel{!}{=} \sum_{i=0}^{l^m-1} b_i \gamma^{i+1}.$$

This is equivalent to $b_i^{\gamma^{-1}} = b_i$ for all $0 \leq i \leq l^m - 1$, i.e.

$$\sum_{j=0}^{v_\chi-1} \beta_{ij}^{\gamma^{-1}} \varepsilon(\eta^{\gamma^{j-1}}) = \sum_{j=0}^{v_\chi-1} \beta_{ij} \varepsilon(\eta^{\gamma^j})$$

and thus, because we can read j modulo v_χ , we have

$$\beta_{i0}^{\gamma^{-1}} \varepsilon(\eta^{\gamma^{v_\chi-1}}) = \beta_{i, v_\chi-1} \varepsilon(\eta^{\gamma^{v_\chi-1}}), \dots, \beta_{i1}^{\gamma^{-1}} \varepsilon(\eta) = \beta_{i0} \varepsilon(\eta). \quad (3.1)$$

This implies that β_{i0} determines b_i .

Analogously, $\gamma^{v_\chi} z \stackrel{!}{=} z \gamma^{v_\chi}$ and $\varepsilon(\eta^{\gamma^{v_\chi}}) = \varepsilon(\eta)$ yield

$$\beta_{i0}^{\gamma^{-v_\chi}} \varepsilon(\eta) = \beta_{i0} \varepsilon(\eta) \quad \forall \quad 0 \leq i \leq l^m - 1. \quad (3.2)$$

Moreover, the central element z commutes with $\varepsilon(\eta)$:

$$\varepsilon(\eta) z \stackrel{!}{=} z \varepsilon(\eta) \iff \sum_{i=0}^{l^m-1} \varepsilon(\eta) b_i \gamma^i \stackrel{!}{=} \sum_{i=0}^{l^m-1} b_i \gamma^i \varepsilon(\eta) = \sum_{i=0}^{l^m-1} b_i \varepsilon(\eta)^{\gamma^{-i}} \gamma^i.$$

This is equivalent to $\varepsilon(\eta) b_i = b_i \varepsilon(\eta)^{\gamma^{-i}} = b_i \varepsilon(\eta^{\gamma^{-i}})$ for all $0 \leq i \leq l^m - 1$. Because the primitive central idempotents $\varepsilon(\eta^{\gamma^{-i}})$ are orthogonal, i.e. $\varepsilon(\eta^{\gamma^{-i}}) \cdot \varepsilon(\eta^{\gamma^{-j}}) \neq 0$ iff $i \equiv j \pmod{v_\chi}$, we get $\varepsilon(\eta) b_i = 0$ and thus

$$b_i = 0 \quad \forall \quad v_\chi \nmid i$$

by $\varepsilon(\eta) b_i = \varepsilon(\eta) \beta_{i0}$ and (3.1). From now on, we therefore consider $z = \sum' b_i \gamma^i$ with $\sum' := \sum_{i=0, v_\chi \nmid i}^{l^m-1}$.

Finally, the central element z commutes with every $b \varepsilon(\eta)$ for $b \in (\mathcal{Q}\Gamma_0)[H]$:

$$b \varepsilon(\eta) z \stackrel{!}{=} z b \varepsilon(\eta) \iff \sum' b \varepsilon(\eta) b_i \gamma^i \stackrel{!}{=} \sum' b_i \gamma^i b \varepsilon(\eta) = \sum' b_i b^{\gamma^{-i}} \varepsilon(\eta) \gamma^i$$

and thus

$$b \varepsilon(\eta) b_i = (b \varepsilon(\eta))(b_i \varepsilon(\eta)) = (b \varepsilon(\eta))(\beta_{i0} \varepsilon(\eta)) = (\beta_{i0} \varepsilon(\eta))(b^{\gamma^{-i}} \varepsilon(\eta)) \quad (3.3)$$

for all $b \in (\mathcal{Q}\Gamma_0)[H]$ and for all $0 \leq i \leq l^m - 1$, $v_\chi \mid i$.

To understand this condition, we first consider $w_\chi \mid i$. In this case, the representation

$$D : (\mathcal{Q}\Gamma_0)[H] \rightarrow (\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1) \times \eta(1)}$$

corresponding to η leads to

$$D(b)D(\beta_{i0}) = D(\beta_{i0})D(b^{\gamma^{-i}}).$$

From the definition of w_χ , we achieve $D(b^{\gamma^{-w_\chi}}) = YD(b)Y^{-1}$ and

$$D(b^{\gamma^{-i}}) = Y^{i/w_\chi}D(b)Y^{-(i/w_\chi)} =: Y_iD(b)Y_i^{-1}$$

with a matrix $Y \in (\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1) \times \eta(1)}$. Observe that Y_i is unique up to central elements of $(\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1) \times \eta(1)}$. Moreover, Y is independent of b because D is multiplicative and thus, even though $\gamma^{w_\chi} \notin (\mathcal{Q}\Gamma_0)[H]$, we see

$$D((bc)^{\gamma^{-w_\chi}}) = D(\gamma^{w_\chi}b\gamma^{-w_\chi}\gamma^{w_\chi}c\gamma^{-w_\chi}) = D(b^{\gamma^{-w_\chi}})D(c^{\gamma^{-w_\chi}}).$$

This yields

$$D(b)D(\beta_{i0}) = D(\beta_{i0})Y_iD(b)Y_i^{-1} \iff D(b)(D(\beta_{i0})Y_i) = (D(\beta_{i0})Y_i)D(b).$$

Since D is surjective, $D(b)$ runs through $(\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1) \times \eta(1)}$ and $D(\beta_{i0})Y_i$ is therefore central in $(\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1) \times \eta(1)}$. Furthermore, there exists a $y_i \in (\mathcal{Q}\Gamma_0)[H]$ with $D(y_i) = Y_i$ and y_i can be chosen as $y_i = y_{w_\chi}^{i/w_\chi}$ because D is an epimorphism. Thus, for every $0 \leq i \leq l^m - 1$ with $w_\chi \mid i$, the element $(\beta_{i0}\varepsilon(\eta))(y_i\varepsilon(\eta))$ maps to a central matrix in $\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0 \cdot \mathbf{1}$.

Because of $\gamma^{v_\chi}\varepsilon(\eta) \in A$, the centre has to be fixed under the action induced by the conjugation by γ^{v_χ} , which by construction is equal to the action of $\sigma_{v_\chi} \in G_0 = \langle \sigma_{v_\chi} \rangle$ on $\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0$, i.e.

$$(\beta_{i0}\varepsilon(\eta))(y_i\varepsilon(\eta)) \mapsto D(\beta_{i0})D(y_i) \in \mathcal{Q}^L\Gamma_0 \cdot \mathbf{1}. \quad (3.4)$$

Next, we show that $b_i = 0$ for $0 \leq i \leq l^m - 1$ with $v_\chi \mid i$ but $w_\chi \nmid i$. Observe that conjugation by γ^{v_χ} induces an automorphism on $(\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta) \cong (\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1) \times \eta(1)}$ because H is normal in G and $\varepsilon(\eta)^{\gamma^{v_\chi}} = \varepsilon(\eta^{\gamma^{v_\chi}}) = \varepsilon(\eta)$. This automorphism fixes the centre of $(\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta)$ because for z a central element and $x \in (\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta)$, we have

$$\gamma^{-v_\chi}z\gamma^{v_\chi}x = \gamma^{-v_\chi}z\gamma^{v_\chi}x\gamma^{-v_\chi}\gamma^{v_\chi} = \gamma^{-v_\chi}\gamma^{v_\chi}x\gamma^{-v_\chi}z\gamma^{v_\chi} = x\gamma^{-v_\chi}z\gamma^{v_\chi}.$$

Thus, it fixes the centre of $(\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1) \times \eta(1)}$, too. But this automorphism does not fix $(\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta) \cong (\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1) \times \eta(1)}$ elementwise; we will call it $c_{\gamma^{v_\chi}}$.

σ_{v_χ} can also be extended to an automorphism on $(\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1)\times\eta(1)}$ via its action on every entry of the matrices in $(\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1)\times\eta(1)}$. This yields the central automorphism $c_{\gamma^{v_\chi}}\sigma_{v_\chi}^{-1}$ of $(\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1)\times\eta(1)}$ which, by the theorem of Skolem-Noether, is the conjugation by a matrix $\tilde{Y} \in (\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1)\times\eta(1)}$.

Now, we can repeat the above calculations for this case. Let σ_i be the automorphism induced by γ^i and observe that $\sigma_i = \sigma_{v_\chi}^{i/v_\chi}$. This time, we get

$$D(b^{\gamma^{-i}})^{\sigma_i} = \tilde{Y}_i D(b) \tilde{Y}_i^{-1} \iff D(b^{\gamma^{-i}}) = \tilde{Y}_i^{\sigma_i^{-1}} D(b)^{\sigma_i^{-1}} \tilde{Y}_i^{\sigma_i^{-1}-1}. \quad (3.5)$$

With $Y_i := \tilde{Y}_i^{\sigma_i^{-1}}$, condition (3.3) yields

$$D(b)D(\beta_{i0}) = D(\beta_{i0})Y_i D(b)^{\sigma_i^{-1}} Y_i^{-1} \iff D(b)(D(\beta_{i0})Y_i) = (D(\beta_{i0})Y_i)D(b)^{\sigma_i^{-1}}. \quad (3.6)$$

Because $b \in (\mathcal{Q}\Gamma_0)[H]$ is arbitrary and D is surjective, we can chose $b \in (\mathcal{Q}\Gamma_0)[H]$ such that $D(b) = \alpha \cdot \mathbf{1}$ is central and $\alpha^{\sigma_i^{-1}} \neq \alpha \in \mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0$. Now, (3.6) yields

$$\alpha \mathbf{1} \cdot D(\beta_{i0})Y_i = D(\beta_{i0})Y_i \alpha^{\sigma_i^{-1}} \mathbf{1} = \alpha^{\sigma_i^{-1}} \mathbf{1} \cdot D(\beta_{i0})Y_i$$

and thus $\alpha \mathbf{1} \cdot D(\beta_{i0}) = \alpha^{\sigma_i^{-1}} \mathbf{1} \cdot D(\beta_{i0})$ by cancelling Y_i . Therefore, we conclude $\alpha x_{\nu\tau} = \alpha^{\sigma_i^{-1}} x_{\nu\tau}$ for every entry $x_{\nu\tau}$ of $D(\beta_{i0})$ and finally $D(\beta_{i0}) = 0$.

These results are now summarized. Let $z = \sum' b_i \gamma^i$ with $b_i = \sum_{j=0}^{v_\chi-1} \beta_{ij} \varepsilon(\eta^{\gamma^j}) \in B$ be a central element of A . We have seen that we can assume $\sum' = \sum_{i=0, w_\chi|i}^{l^m-1}$ and that β_{i0} determines b_i uniquely for each such i .

It remains to show the claimed isomorphism for $Z(A)$. We start with the proof that

$$\begin{aligned} \varphi_1 : Z(A) &\rightarrow \bigoplus' (\mathcal{Q}\Gamma_0)[H] \varepsilon(\eta) \gamma^i, \\ \sum' b_i \gamma^i &\mapsto \sum' b_i \varepsilon(\eta) \gamma^i = \sum' \beta_{i0} \varepsilon(\eta) \gamma^i, \end{aligned}$$

is a monomorphism of $\mathcal{Q}\Gamma_0$ -algebras. First, φ_1 is the identity on $\mathcal{Q}\Gamma_0$. Next, the map φ_1 is injective because $\beta_{i0} \varepsilon(\eta)$ determines b_i by condition (3.1). Additivity of φ_1 being obvious, it remains to check multiplicativity: Let $\sum' b_i \gamma^i$ and $\sum' \tilde{b}_j \gamma^j \in Z(A)$. Then

$$\begin{aligned} \varphi_1 \left(\sum' b_i \gamma^i \right) \cdot \varphi_1 \left(\sum' \tilde{b}_j \gamma^j \right) &= \left(\sum' \beta_{i0} \varepsilon(\eta) \gamma^i \right) \cdot \left(\sum' \tilde{\beta}_{j0} \varepsilon(\eta) \gamma^j \right) \\ &\stackrel{1}{=} \sum' \left(\sum_{\nu+\tau=i} ' \beta_{\nu 0} (\tilde{\beta}_{\tau 0})^{\gamma^{-\nu}} \right) \varepsilon(\eta) \gamma^i. \\ &\stackrel{2}{=} \sum' \left(\sum_{\nu+\tau=i} ' \beta_{\nu 0} \tilde{\beta}_{\tau 0} \right) \varepsilon(\eta) \gamma^i. \end{aligned}$$

For $\stackrel{1}{=}$, we have used the fact that $\varepsilon(\eta)$ is a central idempotent of $(\mathcal{Q}\Gamma_0)[H]$ and $\varepsilon(\eta^{w_\chi}) = \varepsilon(\eta)$; for $\stackrel{2}{=}$ use condition (3.2). Finally

$$\begin{aligned} \varphi_1 \left(\left(\sum' b_i \gamma^i \right) \cdot \left(\sum' \tilde{b}_j \gamma^j \right) \right) &= \varphi_1 \left(\sum' \left(\sum_{\nu+\tau=i} ' b_\nu \tilde{b}_\tau \right) \gamma^i \right) \\ &= \sum' \left(\sum_{\nu+\tau=i} ' b_\nu \tilde{b}_\tau \right) \varepsilon(\eta) \gamma^i \\ &= \sum' \left(\sum_{\nu+\tau=i} ' \beta_{\nu 0} \tilde{\beta}_{\tau 0} \right) \varepsilon(\eta) \gamma^i \end{aligned}$$

and multiplicity is stated.

The action of $\gamma^{w_\chi} = (\gamma^{v_\chi})^{w_\chi/v_\chi}$ on $L = \mathbb{Q}_l(\eta)^{G_0}$ is trivial by the definition of v_χ . It is clearly trivial on $\mathcal{Q}\Gamma_0$, too. This yields the identity

$$\bigoplus' (\mathcal{Q}^L \Gamma_0) \gamma^i = \mathcal{Q}^L \Gamma^{w_\chi}.$$

Next, we fix $y_{w_\chi} \in (\mathcal{Q}\Gamma_0)[H]$ and $y_i = y_{w_\chi}^{i/w_\chi}$ as above and claim that

$$\begin{aligned} \varphi : Z(A) &\rightarrow \bigoplus' (\mathcal{Q}^L \Gamma_0) \gamma^i \cdot \mathbf{1} = \mathcal{Q}^L \Gamma^{w_\chi} \cdot \mathbf{1}, \\ \sum' b_i \gamma^i &\mapsto \sum' D(\beta_{i0}) D(y_i) \gamma^i, \end{aligned}$$

is an isomorphism of $\mathcal{Q}\Gamma_0$ -algebras (compare (3.4)). Again, φ is the identity on $\mathcal{Q}\Gamma_0$. We can write φ as

$$\varphi = D \circ r_y \circ \varphi_1$$

with

$$\begin{aligned} r_y : \bigoplus' (\mathcal{Q}\Gamma_0)[H] \varepsilon(\eta) \gamma^i &\rightarrow \bigoplus' (\mathcal{Q}\Gamma_0)[H] \varepsilon(\eta) \gamma^i, \\ \sum' \beta_{i0} \varepsilon(\eta) \gamma^i &\mapsto \sum' \beta_{i0} \varepsilon(\eta) y_i \varepsilon(\eta) \gamma^i, \end{aligned}$$

a monomorphism of additive groups.

The representation D is here extended to $\bigoplus' (\mathcal{Q}\Gamma_0)[H] \varepsilon(\eta) \gamma^i$ by

$$D \left(\sum' \beta_{i0} \varepsilon(\eta) \gamma^i \right) = \sum' D(\beta_{i0}) \gamma^i,$$

which again is a homomorphism. Observe that, in fact, we only need to extend $D|_{(\mathcal{Q}\Gamma_0)[H] \varepsilon(\eta)}$. Thus, the additivity of φ results from the additivity of φ_1 , r_y and D , the injectivity of φ follows from the injectivity of φ_1 , r_y and $D|_{(\mathcal{Q}\Gamma_0)[H] \varepsilon(\eta)}$.

For multiplicity, we have on the one hand

$$\begin{aligned}
\varphi \left(\sum' b_i \gamma^i \right) \cdot \varphi \left(\sum' \tilde{b}_i \gamma^i \right) &= \left(\sum' D(\beta_{i0}) D(y_i) \gamma^i \right) \cdot \left(\sum' D(\tilde{\beta}_{i0}) D(y_i) \gamma^i \right) \\
&\stackrel{1}{=} \sum' \left(\sum'_{\nu+\tau=i} D(\beta_{\nu 0}) D(y_\nu) D(\tilde{\beta}_{\tau 0}) D(y_\tau) \right) \gamma^i \\
&\stackrel{2}{=} \sum' \left(\sum'_{\nu+\tau=i} D(\beta_{\nu 0}) D(\tilde{\beta}_{\tau 0}) D(y_\nu) D(y_\tau) \right) \gamma^i \\
&\stackrel{3}{=} \sum' \left(\sum'_{\nu+\tau=i} D(\beta_{\nu 0}) D(\tilde{\beta}_{\tau 0}) \right) D(y_i) \gamma^i.
\end{aligned}$$

Here, $\stackrel{1}{=}$ is condition (3.4) together with the fact that $w_\chi \mid i$. For $\stackrel{2}{=}$, we have used that $D(y_\nu) = Y_\nu$ commutes with $D(\beta_{\tau 0})$ by condition (3.2) and the definition of Y_ν . For $\stackrel{3}{=}$, observe that

$$D(y_\nu) D(y_\tau) = Y_{w_\chi^{(\nu+\tau)/w_\chi}} = D(y_{\nu+\tau}) = D(y_i).$$

On the other hand, we keep in mind that \tilde{b}_i and γ^i commute by (3.2) and compute

$$\begin{aligned}
\varphi \left(\left(\sum' b_i \gamma^i \right) \cdot \left(\sum' \tilde{b}_i \gamma^i \right) \right) &= \varphi \left(\sum' \left(\sum'_{\nu+\tau=i} b_\nu \tilde{b}_\tau \right) \gamma^i \right) \\
&= (D \circ r_y) \left(\sum' \left(\sum'_{\nu+\tau=i} \beta_{\nu 0} \tilde{\beta}_{\tau 0} \varepsilon(\eta) \right) \gamma^i \right) \\
&= D \left(\sum' \left(\sum'_{\nu+\tau=i} \beta_{\nu 0} \tilde{\beta}_{\tau 0} \varepsilon(\eta) \right) y_i \varepsilon(\eta) \gamma^i \right) \\
&= \sum' D \left(\sum'_{\nu+\tau=i} \beta_{\nu 0} \tilde{\beta}_{\tau 0} \right) D(y_i) \gamma^i \\
&= \sum' \left(\sum'_{\nu+\tau=i} D(\beta_{\nu 0}) D(\tilde{\beta}_{\tau 0}) \right) D(y_i) \gamma^i.
\end{aligned}$$

Thus, multiplicity is stated.

It remains to prove that φ is surjective. To do so, take a $W_i \in \mathcal{Q}^L \Gamma_0 \cdot \mathbf{1}$ for every $0 \leq i \leq l^m - 1$ with $w_\chi \mid i$. By the surjectivity of D , there exists a $w_i \in (\mathcal{Q} \Gamma_0)[H] \varepsilon(\eta)$ such that $W_i = D(w_i)$. Define $\beta_{i0} \in (\mathcal{Q} \Gamma_0)[H]$ via $D(w_i) = D(\beta_{i0}) D(y_i)$. This β_{i0} determines b_i via (3.1) uniquely and finally $z = \sum' b_i \gamma^i$ maps to $\sum' W_i \gamma^i$. \square

Now, we can compute the structure of A .

Theorem 3.1 *Let A be the simple component of $\mathcal{Q}G$ corresponding to the irreducible $\chi \in R_l G$ and $\sigma_{v_\chi} \in G_0$ with $\langle \sigma_{v_\chi} \rangle = G_0$ and $\eta^{\sigma_{v_\chi}} = \eta^{\gamma^{v_\chi}}$. Then*

- (i) $\dim_{Z(A)} A = \chi(1)^2$,
- (ii) A is split by $\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma^{w_\chi}$,
- (iii) A has Schur index $s_D = \frac{w_\chi}{v_\chi}$ and
- (iv) $A \cong D_{n \times n}$ with $n = \frac{\chi(1)}{s_D}$ and the skew field D is cyclic:

$$D \cong \bigoplus_{i=0}^{w_\chi/v_\chi-1} (\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma^{w_\chi}) \gamma^{v_\chi i} =: (\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma^{w_\chi} / \mathcal{Q}^L \Gamma^{w_\chi}, \sigma_{v_\chi}, \gamma^{w_\chi}).$$

Proof: For (i), let χ' be an irreducible \mathbb{Q}_l^c -character of $St(\eta)$ extending η such that $\chi = \text{ind}_{St(\eta)}^G(\chi')$. Consider $Z_{\chi'}(A) := Z(A)(\chi')$. From

$$\mathbb{Q}_l(\chi') \otimes_{\mathbb{Q}_l} A = (\mathbb{Q}_l(\chi') \otimes_{\mathbb{Q}_l} Z(A)) \otimes_{Z(A)} A \cong (Z_{\chi'}(A) \otimes_{Z(A)} A)^{r_{\chi'}} \quad (3.7)$$

with $r_{\chi'}$ appropriately, we conclude $Z_{\chi'}(A) \cong Z(\mathcal{Q}^{\mathbb{Q}_l(\chi')} G e_\chi)$: First, we show

$$\mathbb{Q}_l(\chi') \otimes_{\mathbb{Q}_l} A = \bigoplus_{\sigma \in \text{Gal}(\chi')}^* (\mathcal{Q}^{\mathbb{Q}_l(\chi')} G) e_{\chi^\sigma} \quad (3.8)$$

with $\text{Gal}(\chi') := G(\mathcal{Q}^{\mathbb{Q}_l(\chi')} \Gamma_0 / \mathcal{Q} \Gamma_0)$ and \bigoplus^* meaning summation modulo type W .

Because χ can be regarded as irreducible character of the finite group G/Γ_0 , there exists a representation of χ over the fields $\mathbb{Q}_l(\chi) \subseteq \mathbb{Q}_l(\chi')$ by [39]. Furthermore, the irreducible constituents η_j of $\text{res}_G^H \chi^\sigma$ are precisely the characters $(\eta^{\gamma^j})^\sigma$ obtained by the irreducible constituents η^{γ^j} of $\text{res}_G^H \chi$. We further recall that the actions of γ and σ commute. Thus,

$$e_{\chi^\sigma} = \sum_{\eta_j | \text{res}_G^H \chi^\sigma} e(\eta_j) = \sum_{j=0}^{w_\chi-1} e(\eta^{\gamma^j})^\sigma = e_\chi^\sigma \in \mathcal{Q}^{\mathbb{Q}_l(\chi')} G$$

because $\mathbb{Q}_l(\eta) \subseteq \mathbb{Q}_l(\chi')$ by construction and e_{χ^σ} is therefore a primitive central idempotent of $\mathcal{Q}^{\mathbb{Q}_l(\chi')} G$, i.e. $\mathcal{Q}^{\mathbb{Q}_l(\chi')} G e_{\chi^\sigma}$ is a simple component of $\mathcal{Q}^{\mathbb{Q}_l(\chi')} G$. Hence summation modulo type W ensures that the summands of the right hand side are distinct because in [32] it is shown that two primitive central idempotents are equal iff the corresponding irreducible characters only differ by a W -twist.

The inclusion ‘ \supseteq ’ of (3.8) is true by the following. From [39], we know that η has a realization over $\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma_0$ and thus over $\mathcal{Q}^{\mathbb{Q}_l(\chi')} \Gamma_0 \supseteq \mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma_0$. This yields that the $(\eta^{\gamma^j})^\sigma$ are absolute irreducible characters of H belonging to Wedderburn components of $\mathbb{Q}_l(\chi') \otimes_{\mathbb{Q}_l} B$, recall that $B = \bigoplus_{j=0}^{v_\chi-1} (\mathcal{Q} \Gamma_0)[H] \varepsilon(\eta^{\gamma^j})$. Therefore, $B e_{\chi^\sigma} \neq 0$ and in particular $A e_{\chi^\sigma} \neq 0$. Finally, $\mathcal{Q}^{\mathbb{Q}_l(\chi')} G e_{\chi^\sigma} \subseteq \mathbb{Q}_l(\chi') \otimes_{\mathbb{Q}_l} A$ follows.

As the Wedderburn components are orthogonal, the direct sum of the $\mathcal{Q}^{\mathbb{Q}_l(\chi')} Ge_{\chi^\sigma}$ is also contained in $\mathbb{Q}_l(\chi') \otimes_{\mathbb{Q}_l} A$ and the claimed inclusion follows.

For the other inclusion, observe that $\mathbb{Q}_l(\chi') \otimes_{\mathbb{Q}_l} A \subseteq \mathcal{Q}^{\mathbb{Q}_l(\chi')} G$ carries a natural $\text{Gal}(\chi')$ -action with fixed points $A \subset \mathcal{Q}G$. Also, $\bigoplus_{\sigma \in \text{Gal}(\chi')}^* (\mathcal{Q}^{\mathbb{Q}_l(\chi')} G) e_{\chi^\sigma}$ carries this action. As the set of fixed points of $\mathcal{Q}^{\mathbb{Q}_l(\chi')} G = \mathbb{Q}_l(\chi') \otimes_{\mathbb{Q}_l} \mathcal{Q}G$ is exactly $\mathcal{Q}G$, the fixed points of the right side make up a two-sided ideal of $\mathcal{Q}G$. Of course, the fixed points of $\bigoplus_{\sigma \in \text{Gal}(\chi')}^* (\mathcal{Q}^{\mathbb{Q}_l(\chi')} G) e_{\chi^\sigma} \subseteq \mathbb{Q}_l(\chi') \otimes_{\mathbb{Q}_l} A$ are contained in the set of fixed points of $\mathbb{Q}_l(\chi') \otimes_{\mathbb{Q}_l} A$, i.e. in A . Since A is simple, the fixed points in $\bigoplus_{\sigma \in \text{Gal}(\chi')}^* (\mathcal{Q}^{\mathbb{Q}_l(\chi')} G) e_{\chi^\sigma}$ are therefore given by A . This implies the equality.

Thus, the simple components on the left and right side of (3.8) coincide. Observe that $Z_{\chi'}(A) \otimes_{Z(A)} A$ is a central simple $Z_{\chi'}(A)$ -algebra because A is a central simple $Z(A)$ -algebra (compare [13, Hilfssatz 14.2]). Together with (3.7), this yields

$$Z_{\chi'}(A) \otimes_{Z(A)} A \cong \mathcal{Q}^{\mathbb{Q}_l(\chi')} Ge_{\chi^\sigma}.$$

Now, we compute the centres of the simple components of (3.8). On the right side, the centres are obviously the $Z(\mathcal{Q}^{\mathbb{Q}_l(\chi')} Ge_{\chi^\sigma})$. The simple components of the left side are, by (3.7), isomorphic to the central simple $Z_{\chi'}(A)$ -algebra $Z_{\chi'}(A) \otimes_{Z(A)} A$. Thus, (3.8) implies the claimed isomorphism

$$Z_{\chi'}(A) \cong Z(\mathcal{Q}^{\mathbb{Q}_l(\chi')} Ge_{\chi^\sigma}).$$

In [32, p. 555], it is shown that $\dim_{Z(\mathcal{Q}^{\mathbb{Q}_l(\chi')} Ge_{\chi^\sigma})} \mathcal{Q}^{\mathbb{Q}_l(\chi')} Ge_{\chi^\sigma} = \chi(1)^2$ and thus

$$\dim_{Z(A)} A = \dim_{Z_{\chi'}(A)} (Z_{\chi'}(A) \otimes_{Z(A)} A) = \chi(1)^2$$

results.

We turn to the proof of (iv). The theorem of Wedderburn implies that $A \cong D_{n \times n}$ is a full matrix ring over a skew field D . The dimension of D over the centre of A is the square s_D^2 with s_D being the Schur index of A . Thus

$$\chi(1)^2 = \dim_{Z(A)} A = s_D^2 n^2.$$

For the computation of D , we use the fact that $A \cong E^n$ for a minimal right ideal $E = \varepsilon A$ of A and $D \cong \varepsilon A \varepsilon$ with ε a primitive idempotent of A . Analogously, there exists a primitive idempotent ε_1 of $(\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta)$ for the minimal right ideal $S_\eta = \varepsilon_1(\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta)$ of $(\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta)$. We have seen that $(\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta)$ is the full matrix ring $(\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1) \times \eta(1)}$ and therefore $\varepsilon_1(\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta)\varepsilon_1 \cong \mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0$ with

$$\varepsilon_1 \mapsto \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in (\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1) \times \eta(1)}.$$

With $R := (\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1) \times \eta(1)} \cong (\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta)$, we get

$$\begin{aligned} \bigoplus_{i=0}^{l^m-1} (S_\eta)\gamma^i &= \bigoplus_{i=0}^{l^m-1} (\varepsilon_1 R)\gamma^i = \bigoplus_{i=0}^{l^m-1} \left(\underbrace{(\varepsilon_1, 0, \dots, 0)}_{v_\chi} \bigoplus_{j=0}^{v_\chi-1} R \right) \gamma^i \\ &= (\varepsilon_1, 0, \dots, 0) \bigoplus_{i=0}^{l^m-1} \left(\bigoplus_{j=0}^{v_\chi-1} R \right) \gamma^i \\ &\cong (\varepsilon_1, 0, \dots, 0) \bigoplus_{i=0}^{l^m-1} \left(\bigoplus_{j=0}^{v_\chi-1} (\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta^{\gamma^j}) \right) \gamma^i \end{aligned}$$

and furthermore

$$\begin{aligned} \varepsilon_1 A \varepsilon_1 &\cong (\varepsilon_1, 0, \dots, 0) \left(\bigoplus_{i=0}^{l^m-1} \left(\bigoplus_{j=0}^{v_\chi-1} R \right) \gamma^i \right) (\varepsilon_1, 0, \dots, 0) \\ &= \bigoplus_{i=0}^{l^m-1} \left((\varepsilon_1, 0, \dots, 0) \bigoplus_{j=0}^{v_\chi-1} R(\varepsilon_1^{\gamma^{-i}}, 0, \dots, 0) \right) \gamma^i \\ &= \bigoplus_{i=0}^{l^m-1} (\varepsilon_1 R \varepsilon_1^{\gamma^{-i}}) \gamma^i \stackrel{1}{=} \bigoplus_{i=0, v_\chi \mid i}^{l^m-1} (\varepsilon_1 R \varepsilon_1^{\gamma^{-i}}) \gamma^i. \end{aligned}$$

For $\stackrel{1}{=}$, we have used that conjugation by γ^i permutes the Wedderburn components of $(\mathcal{Q}\Gamma_0)[H]$, i.e. $\varepsilon_1^{\gamma^{-i}} \in (\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta^{\gamma^{-i}}) = (\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta^{\gamma^{v_\chi-i}})$, and thus only the conjugations by γ^i , $v_\chi \mid i$, yield nonzero summands. We have seen in (3.5) that

$$D(\varepsilon_1^{\gamma^{-i}}) = Y_i D(\varepsilon_1)^{\sigma_i^{-1}} Y_i^{-1}$$

Therefore, $D(\varepsilon_1^{\gamma^{-i}})$ (resp. $\varepsilon_1^{\gamma^{-i}}$) is still an idempotent of a minimal right ideal of $(\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0)_{\eta(1) \times \eta(1)} \cong (\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta)$. Furthermore, it is a primitive idempotent because otherwise $(Y_i^{-1} D(\varepsilon_1^{\gamma^{-i}}) Y_i)^{\sigma_i} = D(\varepsilon_1)$ is not primitive, a contradiction.

As primitive idempotent, $\varepsilon_1^{\gamma^{-i}}$ is a matrix of rank 1 and thus

$$\varepsilon_1 R \varepsilon_1^{\gamma^{-i}} = \left\{ \left(\begin{pmatrix} a_1 & \dots & a_{\eta(1)} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \right) \alpha : \alpha \in \mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0 \right\}$$

with at least one of the $a_\nu \neq 0$ for $1 \leq \nu \leq \eta(1)$. This implies $\varepsilon_1 R \varepsilon_1^{\gamma^{-i}} \cong \mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0$ as additive groups and also

$$\varepsilon_1 A \varepsilon_1 \cong \bigoplus_{i=0, v_\chi \mid i}^{l^m-1} \mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma_0 \gamma^i = \bigoplus_{i=0, v_\chi \mid i}^{w_\chi-1} \mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma^{w_\chi} \gamma^i =: A_0$$

as additive groups. Concerning multiplication in $\varepsilon_1 A \varepsilon_1$, observe

$$\begin{aligned} \varepsilon_1 R \varepsilon_1^{\gamma^{-i}} \gamma^i \cdot \varepsilon_1 R \varepsilon_1^{\gamma^{-i'}} \gamma^{i'} &\stackrel{1}{=} \varepsilon_1 R \varepsilon_1^{\gamma^{-i}} \gamma^i R \varepsilon_1^{\gamma^{-i'}} \gamma^{i'} \\ &\stackrel{2}{=} \varepsilon_1 R \varepsilon_1^{\gamma^{-i}} R \varepsilon_1^{\gamma^{-(i+i')}} \gamma^{i+i'} \stackrel{3}{=} \varepsilon_1 R \varepsilon_1^{\gamma^{-(i+i')}} \gamma^{i+i'} \end{aligned}$$

and $\stackrel{1}{=}$ is true because $\varepsilon_1^{\gamma^{-i}}$ is an idempotent, $\stackrel{2}{=}$ follows by $R \cong (\mathcal{Q}\Gamma_0)[H]\varepsilon(\eta)$: As H is normal in G , conjugation by $\gamma^{(-i)}$ fixes the algebra $(\mathcal{Q}\Gamma_0)[H]$. Furthermore, $v_\chi \mid i$ ensures that even the Wedderburn components of $(\mathcal{Q}\Gamma_0)[H]$ are fixed, i.e. $R^{\gamma^{-i}} = R$. For $\stackrel{3}{=}$ note that R is simple; thus, the two-sided ideal $R \varepsilon_1^{\gamma^{-(i+i')}} R$ is the whole ring R .

The direct i -sum in the upper description of $\varepsilon_1 A \varepsilon_1$ is hence not orthogonal and the multiplication rules on $\varepsilon_1 A \varepsilon_1$ and on A_0 coincide. Therefore, $\varepsilon_1 A \varepsilon_1 \cong A_0$ as $\mathcal{Q}\Gamma_0$ -algebras and $\varepsilon_1 A \varepsilon_1$ is the claimed crossed product

$$\varepsilon_1 A \varepsilon_1 \cong (\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma^{w_\chi} / \mathcal{Q}^L \Gamma^{w_\chi}, \sigma_{v_\chi}, \gamma^{w_\chi}).$$

For $\varepsilon_1 A \varepsilon_1$ to be a skew field, it remains to prove that the Schur index s of the crossed product is equal to $[\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma^{w_\chi} : \mathcal{Q}^L \Gamma^{w_\chi}] = [\mathbb{Q}_l(\eta) : L] = w_\chi / v_\chi$. In this case, $\varepsilon_1 A \varepsilon_1 \cong D$ is the skew field underlying A and $s_D = s = w_\chi / v_\chi$.

To show this, let N denote the norm of the field extension $\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma^{w_\chi} / \mathcal{Q}^L \Gamma^{w_\chi}$ and set $o(\gamma^{w_\chi})$ the order of γ^{w_χ} in $(\mathcal{Q}^L \Gamma^{w_\chi})^\times / N((\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma^{w_\chi})^\times)$. We use the fact that the crossed product A_0 is a skew field if $o(\gamma^{w_\chi})$ equals the degree $[\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma^{w_\chi} : \mathcal{Q}^L \Gamma^{w_\chi}]$ (see [28, (30.7)]). Then, the Schur index is $s = [\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma^{w_\chi} : \mathcal{Q}^L \Gamma^{w_\chi}] = w_\chi / v_\chi$ as claimed.

For an upper bound of $o(\gamma^{w_\chi})$, we compute $N(\gamma^{w_\chi}) = (\gamma^{w_\chi})^{w_\chi / v_\chi}$. This implies that $o(\gamma^{w_\chi})$ divides $w_\chi / v_\chi =: l^r$ and thus $s \leq w_\chi / v_\chi$. To compute a lower bound, we first identify

$$\mathcal{Q}^{\mathbb{Q}_l(\eta)} \Gamma^{w_\chi} \cong \text{Quot}(\mathbb{Z}_l[\eta][[T]]), \quad \gamma^{w_\chi} \leftrightarrow 1 + T.$$

Now, assume that the order of γ^{w_χ} is less than w_χ / v_χ , i.e. that there exists an $a \in \text{Quot}(\mathbb{Z}_l[\eta][[T]])$ with $N(a) = (1 + T)^{l^t}$ where $l^t < w_\chi / v_\chi = l^r$. By definition, $a = \frac{f(T)}{g(T)}$ with $f(T), g(T) \in \mathbb{Z}_l[\eta][[T]]$. Let ℓ generate the maximal ideal of $\mathbb{Z}_l[\eta]$ above l . Then, the Weierstraß preparation theorem implies that

$$a = \frac{\ell^{n_1} F(T) u_1}{\ell^{n_2} G(T) u_2}$$

with $F(T)$ and $G(T)$ distinguished polynomials in $\mathbb{Z}_l[\eta][[T]]$ and u_1 and u_2 units in $(\mathbb{Z}_l[\eta][[T]])^\times$. The norm N is now the Galois norm of the field extension $\mathbb{Q}_l(\eta)/L$ and thus $N(\ell)$ generates the maximal ideal of the ring of integers \mathfrak{o}_L of L because the extension is totally ramified. Furthermore, the norms of the distinguished polynomials resp. units are distinguished polynomials resp. units in $\mathfrak{o}_L[[T]]$. Applying N

to a thus yields

$$(1+T)^{l^t} = N(a) = \frac{N(\ell)^{n_1} N(F(T)) N(u_1)}{N(\ell)^{n_2} N(G(T)) N(u_2)}$$

and therefore, with $u := u_1/u_2 \in (\mathbb{Z}_l[\eta][[T]])^\times$, we see

$$N(\ell)^{n_2} N(G(T)) (1+T)^{l^t} = N(\ell)^{n_1} N(F(T)) N(u).$$

Because $(1+T)^{l^t}$ is a unit, the Weierstraß preparation theorem for $\mathfrak{o}_L[[T]]$ now implies that $n_1 = n_2$, $N(F(T)) = N(G(T))$, and thus that $F(T)$ and $G(T)$ only differ by a unit in $(\mathbb{Z}_l[\eta][[T]])^\times$. We conclude that $a \in (\mathbb{Z}_l[\eta][[T]])^\times$.

We set $a = \sum_{i=0}^{\infty} a_i T^i$ with $a_i \in \mathbb{Z}_l[\eta]$ for all $i \geq 1$ and $a_0 \in (\mathbb{Z}_l[\eta])^\times$, thus

$$\begin{aligned} (1+T)^{l^t} &= 1 + l^t T + \dots + T^{l^t} = \prod_{\sigma \in G(\mathbb{Q}_l(\eta)/L)} \left(\sum_{i=0}^{\infty} a_i T^i \right)^\sigma \\ &= \prod_{\sigma \in G(\mathbb{Q}_l(\eta)/L)} \left(\sum_{i=0}^{\infty} a_i^\sigma T^i \right) = N(a_0) + a_0 \text{Tr}(a_1) T + \dots \end{aligned}$$

with Tr the trace of the field extension $\mathbb{Q}_l(\eta)/L$. Comparing the coefficients on both sides, we first see that $N(a_0) = 1$ and therefore we can assume $a_0 = 1$ without loss of generality (otherwise divide a by $a_0 \neq 0$, this new a has the same norm as the old one). Next, we consider the condition $l^t = \text{Tr}(a_1)$. This equation is now to be read in $\mathbb{Q}_l(\eta)/L$. Set $\mathbb{Q}_l(\eta) =: \mathbb{Q}_l(\zeta)$ with ζ a primitive l -power root of unity. We achieve $a_1 = \alpha_0 + \alpha_1 \zeta + \dots + \alpha_{l^r-1} \zeta^{l^r-1}$ with $\alpha_i \in \mathfrak{o}_L$ for $0 \leq i < l^r$, recall $l^r = [\mathbb{Q}_l(\eta) : L]$. Thus

$$\begin{aligned} l^t &= \text{Tr}(a_1) = \text{Tr}(\alpha_0 + \alpha_1 \zeta + \dots + \alpha_{l^r-1} \zeta^{l^r-1}) \\ &= l^r \alpha_0 + \alpha_1 \text{Tr}(\zeta) + \dots + \alpha_{l^r-1} \text{Tr}(\zeta^{l^r-1}). \end{aligned}$$

Next, we compute the trace of the powers of ζ . For this, we look more closely at the field extension $\mathbb{Q}_l(\zeta)/L$. As $\mathbb{Q}_l(\zeta)$ is cyclic over \mathbb{Q}_l , we conclude that L itself is a cyclotomic field $L = \mathbb{Q}_l(\xi)$ with $\xi = \zeta^{l^r}$. Thus, the minimal polynomial of ζ over L is $p(X) = X^{l^r} - \xi$. This implies that $\text{Tr}_{\mathbb{Q}_l(\zeta)/L}(\zeta) = 0$. Analogously, we obtain

$$\text{Tr}_{\mathbb{Q}_l(\zeta^i)/L}(\zeta^i) = 0 \text{ for all } 0 \leq i < l^r$$

and furthermore, we set $i = l^\nu \iota$ with $0 \leq \iota < l$ for all $0 \leq i < l^r$. Obviously, we have $\mathbb{Q}_l(\zeta^i) = \mathbb{Q}_l(\zeta^{l^\nu})$. With this, we compute

$$\begin{aligned} \text{Tr}(\zeta^i) &= \text{Tr}_{\mathbb{Q}_l(\zeta)/L}(\zeta^i) = \text{Tr}_{\mathbb{Q}_l(\zeta^i)/L} \circ \text{Tr}_{\mathbb{Q}_l(\zeta)/\mathbb{Q}_l(\zeta^i)}(\zeta^i) \\ &= \text{Tr}_{\mathbb{Q}_l(\zeta^{l^\nu})/L}(l^\nu \zeta^i) = l^\nu \text{Tr}_{\mathbb{Q}_l(\zeta^i)/L}(\zeta^i) = 0. \end{aligned}$$

Therefore, we finally see

$$l^t = \text{Tr}(a_1) = l^r \alpha_0, \text{ i.e. } \alpha_0 = l^{t-r} \in \mathbb{Z}_l[\eta].$$

This is not possible for $t < r$, and we thus have a contradiction to the assumption that the order of γ^{w_χ} was smaller than w_χ/v_χ .

We obtain $s \geq w_\chi/v_\chi$ and finally $s_D = s = w_\chi/v_\chi$. This shows both (iii) and (iv).

For the claim (ii), we have seen that $\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma^{w_\chi}$ is a maximal subfield of the cyclic skew field D and thus is a splitting field of D and A (see e.g. [28, Thm (30.8)]). \square

Corollary 3.1 *The $\mathcal{Q}^c\Gamma_0$ -algebra \mathcal{Q}^cG splits.*

Finally, we give an example of a Galois group G which causes nontrivial Schur indices¹:

Let $l = 3$ and $H = \langle h \rangle$ be the cyclic group of order 9. $G = H \rtimes \Gamma$ is determined by the action $h^\gamma = h^4$. Define the absolute irreducible constituent η of $\text{res}_G^H \chi$ by $\eta(h) = \zeta_9$ with ζ_9 a primitive ninth root of unity, thus $St(\eta) = H \times \langle \gamma^3 \rangle$ and $\chi = \text{ind}_{St(\eta)}^G \chi'$ with $\chi'(h) = \eta(h)$ and $\chi'(\gamma^3) = 1$.

Then, $\chi(1) = 3$, $w_\chi = 3$ and $\text{Gal} = G(\mathbb{Q}_l(\eta)/\mathbb{Q}_l) = G(\mathbb{Q}_l(\zeta_9)/\mathbb{Q}_l) = \langle \sigma \rangle$ with $\eta(h)^\sigma = \zeta_9^\sigma = \zeta_9^2$. Therefore, the action of γ on $\eta(H)$ can be expressed as Galois action: $\eta(h)^\gamma = \eta(h)^4 = \eta(h)^{\sigma^2}$ which implies $v_\chi = 1$ and $G_0 = \langle \sigma^2 \rangle$.

We achieve that the simple component A of $\mathcal{Q}G$ determined by χ has Schur index $s_D = w_\chi/v_\chi = 3$ and dimension $\dim_{Z(A)} A = \chi(1)^2 = 9 = s_D^2$ and is therefore the cyclic skew field

$$D = (\mathcal{Q}^{\mathbb{Q}_l(\zeta_9)}\Gamma^3 / \mathcal{Q}^{\mathbb{Q}_l(\zeta_3)}\Gamma^3, \sigma_{v_\chi} = \sigma^2, \gamma^3)$$

with ζ_3 a primitive third root of unity.

This example can be realized as Galois group over \mathbb{Q} according to the situation of the Iwasawa extension examined earlier: First, we show that $\mathbb{Z}/9 \rtimes \mathbb{Z}/3$ can be realized as Galois group over \mathbb{Q} . With this realization, we then construct a field extension of \mathbb{Q} with Galois group G .

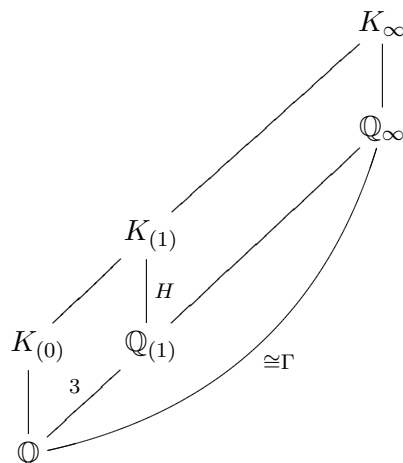
Clearly, $\mathbb{Q}(\zeta_9 + \zeta_9^{-1}) = \mathbb{Q}_{(1)} \subsetneq \mathbb{Q}_\infty$ is a subfield of the cyclotomic \mathbb{Z}_3 -extension $\mathbb{Q}_\infty/\mathbb{Q}$ and $\mathbb{Z}/3 \cong G(\mathbb{Q}(\zeta_9 + \zeta_9^{-1})/\mathbb{Q})$. For the whole group $\mathbb{Z}/9 \rtimes \mathbb{Z}/3$, observe that in [23] this is formulated as imbedding problem $\mathcal{E}_{\mathbb{Z}/3}(G(\mathbb{Q}^c/\mathbb{Q}), \mathbb{Z}/9 \rtimes \mathbb{Z}/3)$ corresponding to the diagram

$$\begin{array}{ccc} & & G(\mathbb{Q}^c/\mathbb{Q}) \\ & & \downarrow \varphi \\ \mathbb{Z}/9 \rtimes \mathbb{Z}/3 & \xrightarrow{f} & \mathbb{Z}/3 \end{array}$$

with epimorphisms f and φ . The problem is whether there exists a surjective homomorphism $\psi : G(\mathbb{Q}^c/\mathbb{Q}) \rightarrow \mathbb{Z}/9 \rtimes \mathbb{Z}/3$ with $f \circ \psi = \varphi$. This is shown to

¹A first such example was given by A. Weiss (unpublished).

be equivalent to the existence of a Galois extension $K_{(1)} \supseteq \mathbb{Q}(\zeta_9 + \zeta_9^{-1}) \supseteq \mathbb{Q}$ with Galois group $G(K_{(1)}/\mathbb{Q}) \cong \mathbb{Z}/9 \rtimes \mathbb{Z}/3$ such that the canonical projection $G(K_{(1)}/\mathbb{Q}) \rightarrow G(\mathbb{Q}(\zeta_9 + \zeta_9^{-1})/\mathbb{Q})$ coincides with $\mathbb{Z}/9 \rtimes \mathbb{Z}/3 \rightarrow \mathbb{Z}/3$.



An affirmative answer to this question is given by [23, Cor 6]. To apply this, note that the field $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$ contains $m(K_{(1)}) = 2$ roots of unity and thus $m(K_{(1)})$ is prime to the exponent of the kernel of f . Now, we see that this imbedding problem is solvable because every group extension $\mathbb{Z}/9 \rtimes \mathbb{Z}/3 \rightarrow \mathbb{Z}/3$ splits.

For the realization of $G = H \rtimes \Gamma \cong \mathbb{Z}/9 \rtimes \Gamma$, consider the diagram with $\mathbb{Q}_{(1)} = \mathbb{Q}(\zeta_9 + \zeta_9^{-1})$. By the above defined action of Γ on H , we see that K_{∞} is abelian over $\mathbb{Q}_{(1)}$ and therefore $G(K_{\infty}/\mathbb{Q}_{\infty}) \cong H$ and $G(K_{\infty}/\mathbb{Q}) \cong H \rtimes \Gamma$ as

desired. Finally, note that $K_{(1)}$ is of odd index over \mathbb{Q} and thus the constructed realization of G is totally real.

Remark 3.1 With Chapter 2 in mind, there is the natural question whether the structural results achieved in the present chapter can be generalized to non-pro- l groups. However, already the split case $G = H \times \Gamma$ indicates that this is not possible. When H is an l -group, the Schur indices of the Wedderburn components to all irreducible \mathbb{Q}_l^c -characters χ of G with open kernel are trivial, because in this case $w_{\chi} = 1$. But if H is not an l -group, then the Schur index s_D of the component corresponding to χ is essentially only restricted by $s_D \mid \chi(1)$ (compare e.g. [19]).

4 Reduction II: Uniqueness

The aim of this chapter is to reduce the conjecture of Suslin for the algebra $\mathcal{Q}G$ for profinite groups $G = H \rtimes \Gamma$ to the pro- l case. More precisely, we assume that $SK_1(\mathcal{Q}^N U) = 1$ for pro- l groups U and finite unramified extensions N/\mathbb{Q}_l to show that $\mathcal{Q}G$ has trivial reduced Whitehead group.

First, we cite the following theorem (see [33, p. 167]). For this, recall that, for a prime number $q \neq l$, G is a \mathbb{Q}_l - q -elementary group if $G = H \times \Gamma$ with Γ a central open subgroup of G isomorphic to Γ_k and H a finite \mathbb{Q}_l - q -elementary group; i.e. $H = \langle s \rangle \rtimes H_q$ is the semidirect product of a cyclic group $\langle s \rangle$ of order prime to q and a q -group H_q whose action on $\langle s \rangle$ induces a homomorphism $H_q \rightarrow G(\mathbb{Q}_l(\beta)/\mathbb{Q}_l)$. Here, β is a primitive root of unity of order $|\langle s \rangle|$. For $q = l$, the group G is called \mathbb{Q}_l - l -elementary if $G = \langle s \rangle \rtimes U$ is the semidirect product of a finite cyclic group $\langle s \rangle$ of order prime to l and an open pro- l subgroup U whose action on $\langle s \rangle$ induces a homomorphism $U \rightarrow G(\mathbb{Q}_l(\beta)/\mathbb{Q}_l)$ with again β a primitive root of unity of order $|\langle s \rangle|$.

Theorem 4.1 (Ritter, Weiss) *Let $G = H \rtimes \Gamma$ and q be a prime number which might be equal to l . Then, $SK_1(\mathcal{Q}G) = 1$ if $SK_1(\mathcal{Q}G') = 1$ for all open \mathbb{Q}_l - q -elementary subgroups G' of G .*

Thus, we have to compute $SK_1(\mathcal{Q}G)$ for \mathbb{Q}_l - q -elementary groups G .

We begin with the case $q \neq l$; in particular, G is a direct product $G = H \times \Gamma$. Thus, G fulfils condition (i) of Section 1.3 and therefore $SK_1(\mathcal{Q}G) = 1$ follows immediately.

Proposition 4.1 *Let $q \neq l$ be a prime number and $G = H \times \Gamma$ a \mathbb{Q}_l - q -elementary group. Then*

$$SK_1(\mathcal{Q}G) = 1.$$

□

Next, we consider the case $q = l$, i.e. $G = \langle s \rangle \rtimes U$ with a finite cyclic group $\langle s \rangle$ of order prime to l and U an open pro- l subgroup.

As in Chapter 2, we fix a finite set $\{\beta_i\}$ of representatives of the $G(\mathbb{Q}_l^c/\mathbb{Q}_l)$ -orbits of the irreducible \mathbb{Q}_l^c -characters of $\langle s \rangle$. Again, we set $N_i := \mathbb{Q}_l(\beta_i)$ and $\mathfrak{o}_i := \mathbb{Z}_l[\beta_i]$. We

denote the stabilizer group of β_i by $U_i := \{u \in U : \beta_i^u = \beta_i\}$. Clearly, $U_i \triangleleft U$ and $A_i := U/U_i \leq G(N_i/\mathbb{Q}_l)$. Thus, A_i is cyclic because N_i is unramified over \mathbb{Q}_l . We fix a representative $x_i \in U$ with $\langle \overline{x_i} \rangle = U/U_i = A_i$. Finally, we set $G_i := \langle s \rangle \rtimes U_i$.

We next read the structure of $\mathcal{Q}G$ in these terms. For this, recall that the

$$e_i := \frac{1}{|\langle s \rangle|} \sum_{\nu \bmod |\langle s \rangle|} \text{tr}_{N_i/\mathbb{Q}_l}(\beta_i(s^{-\nu})) s^\nu \in \mathbb{Z}_l \langle s \rangle$$

are the primitive central idempotents of the group algebra $\mathbb{Q}_l \langle s \rangle$, and furthermore they are central idempotents of $\mathcal{Q}G$. Because the e_i are orthogonal in $\mathbb{Q}_l \langle s \rangle$, we have $e_i e_j = 0$ for $i \neq j$ in $\mathcal{Q}G$, too. Therefore, we conclude $\bigoplus_i e_i \mathcal{Q}G \subseteq \mathcal{Q}G$. For

$$\mathcal{Q}G = \bigoplus_i e_i \mathcal{Q}G,$$

it remains to show the other inclusion $\mathcal{Q}G \subseteq \bigoplus_i e_i \mathcal{Q}G$. We use that $\sum_i e_i = 1$ is true in $\mathbb{Q}_l \langle s \rangle$ and therefore it is true in $\mathcal{Q}G$, too. Thus, $\mathcal{Q}G = 1 \cdot \mathcal{Q}G \subseteq \bigoplus_i e_i \mathcal{Q}G$. We are now ready to state

Lemma 4.1 *With the above notations, we have:*

- (i) $e_i \mathcal{Q}G_i \cong \mathbb{Q}_l(\beta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i = \mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i$,
- (ii) $e_i \mathcal{Q}G \cong \bigoplus_{j=0}^{l^n-1} (\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i) x^j =: (\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i) \star \langle x \rangle$,
where x acts on U_i by conjugation and on $\mathbb{Q}_l(\beta_i)$ via τ .

Proof: (i) is stated in [33, p. 160] and (ii) follows immediately by the definition of the crossed product. \square

Proposition 4.2 *With the above notations, assume $SK_1(\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i) = 1$ for all characters β_i of $\langle s \rangle$. Then, the following are equivalent:*

- (i) $SK_1(\mathcal{Q}G) = 1$.
- (ii) $SK_1((\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i) \star \langle x \rangle) = 1$ for all characters β_i of $\langle s \rangle$.

Proof: This follows immediately by $\mathcal{Q}G = \bigoplus_i e_i \mathcal{Q}G$ and Lemma 4.1. \square

From now on, we fix the primitive root of unity β_i with $\beta_i(s) = \beta_i$ and a representative $x_i \in U$ such that $\langle \overline{x_i} \rangle = U/U_i$. Then, $\overline{x_i}$ maps to some τ_i under the injection $U/U_i \hookrightarrow G(\mathbb{Q}_l(\beta_i)/\mathbb{Q}_l) = G(N_i/\mathbb{Q}_l)$. The order of $\overline{x_i}$ clearly is a power of l , we set $l^n := |U/U_i|$. Thus, we see $x_i^{l^n} \in U_i$. Although n depends on i , we omit this in the notation. Moreover, for the sake of brevity, we set $x := x_i$ and $\tau := \tau_i$, but still keep in mind the underlying β_i .

As the structure $\mathcal{Q}U_i$ is well known from Chapter 3, we now examine $(\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i) \star \langle x \rangle$. Because $(\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i) \star \langle x \rangle$ is isomorphic to $e_i \mathcal{Q}G$, this algebra is semisimple.

Let W' be the Wedderburn component of $\mathcal{Q}U_i$ corresponding to $\chi \in R_l U_i$ and set

$$W = \mathbb{Q}_l(\beta_i) \otimes_{\mathbb{Q}_l} W' = (\mathbb{Q}_l(\beta_i) \otimes_{\mathbb{Q}_l} Z(W')) \otimes_{Z(W')} W' \subseteq \mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i.$$

As $\mathbb{Q}_l(\beta_i)$ and $F' := Z(W') = \mathcal{Q}^L \Gamma^{w_\chi}$ are linearly disjoint over \mathbb{Q}_l , the tensor product $\mathbb{Q}_l(\beta_i) \otimes_{\mathbb{Q}_l} F'$ is a field and thus W is still a simple algebra and therefore a Wedderburn component of $\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i$ with centre $F := Z(W) = \mathbb{Q}_l(\beta_i) \otimes F'$.

Then, x acts on W as it acts on $\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i$. This action fixes the algebra $\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i$ as a whole, but might not fix W . If $W^x \neq W$, then W^x is another Wedderburn component of $\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i$ by the following: W^x is a two-sided ideal of $\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i$ because W is a two-sided ideal of $\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i$. Furthermore, it has centre F^x with $F = Z(W)$. As seen above, $F = \mathbb{Q}_l(\beta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}^L \Gamma^{w_\chi}$ is a field and therefore F^x is a field, too. But as a semisimple algebra with a field as centre, W^x is already a simple algebra. Thus, x permutes the Wedderburn components of $\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i$ and $W^x \cdot W = 0$ if $W^x \neq W$ because of the orthogonality of Wedderburn components.

Note that the minimal j , such that $W^{x^j} = W$, is an l -power because this is the length of the orbit of W in the set of Wedderburn components of $\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i$ under the action of $\langle \bar{x} \rangle$.

Proposition 4.3 *Let W be a simple component of $\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i$ with centre F . Set $0 \leq d \leq n$ to be minimal such that $W^{x^{l^d}} = W$. Then,*

$$\tilde{W} := \bigoplus_{j=0}^{l^d-1} (W^{x^j} \oplus W^{x^j} x \oplus \dots \oplus W^{x^j} x^{l^n-1})$$

is a simple component of $(\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i) \star \langle x \rangle$ with centre $Z(\tilde{W}) = F^{\langle x^{l^d} \rangle} =: E$. Furthermore, \tilde{W} is the full matrix ring

$$\tilde{W} = V_{l^d \times l^d} \quad \text{with} \quad V := W \oplus W x^{l^d} \oplus \dots \oplus W x^{l^d(l^n-d-1)}.$$

Proof: We set $y := x^{l^d}$ and $m := n - d$, i.e. $y^{l^m} = x^{l^n} \in U_i$.

First, \tilde{W} is a two-sided ideal of $(\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i) \star \langle x \rangle$; for this, we only have to check that it is closed under multiplication with x , which is obvious. Thus it is the direct sum of some Wedderburn components.

Next, we show that the centre of \tilde{W} is a field, which automatically implies that \tilde{W} is a simple algebra. We start with the computation of the centre $Z(V)$ of V . Here, we do not consider the trivial case $d = n$, i.e. $V = W$ and therefore $Z(V) = Z(W) = F$ is a field.

We assume $0 \leq d < n$ and take an element $z = w_0 + w_1y + \dots + w_{l^m-1}y^{l^m-1} \in Z(V)$. For any $w \in W \subseteq V$, we see

$$\begin{aligned} zw &= w_0w + w_1w^{y^{-1}}y + \dots + w_{l^m-1}w^{y^{-(l^m-1)}}y^{l^m-1}, \\ wz &= ww_0 + ww_1y + \dots + ww_{l^m-1}y^{l^m-1}. \end{aligned}$$

Because $zw = wz$, we conclude that $w_0 \in Z(W) = F$ and

$$w_1w^{y^{-1}} = ww_1, \dots, w_{l^m-1}w^{y^{-(l^m-1)}} = ww_{l^m-1}. \quad (4.1)$$

Assume for the moment that $w \in Z(W) = F$. Then, (4.1) implies $w_1w^{y^{-1}} = w_1w, \dots, w_{l^m-1}w^{y^{-(l^m-1)}} = w_{l^m-1}w$. But as $F = \mathbb{Q}_l(\beta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}^L \Gamma^{w_\chi}$, we can specialize to $w = \beta_i$. By definition, y does not act trivial on β_i (otherwise $y \in U_i$) and thus $w_1 = \dots = w_{l^m-1} = 0$.

Moreover, z fulfils $yz = zy$. As we have already seen that $z = w_0 \in F$, this implies that $z \in F^{\langle y \rangle}$. Thus $Z(V) \subseteq F^{\langle y \rangle}$. Because the other inclusion $Z(V) \supseteq F^{\langle y \rangle}$ is trivially true, we finally conclude

$$Z(V) = F^{\langle y \rangle}.$$

Now, we are ready to show that $Z(\tilde{W}) = Z(V) = F^{\langle y \rangle}$. For the rest of the proof, we will again allow the trivial case, i.e. $0 \leq d \leq n$. We use the relation

$$\begin{aligned} \tilde{W} &= \bigoplus_{j=0}^{l^d-1} (W^{x^j} \oplus W^{x^j}x \oplus \dots \oplus W^{x^j}x^{l^n-1}) \\ &= \bigoplus_{j=0}^{l^d-1} (V^{x^j} \oplus V^{x^j}x \oplus \dots \oplus V^{x^j}x^{l^d-1}). \end{aligned}$$

Let $0 \leq j \leq l^d - 1$. Because $W^{x^j} \neq W$, we have seen $W^{x^j} \cdot W = 0$ and therefore $V^{x^j} \cdot V = 0$.

We choose $z = \sum_{i,j=0}^{l^d-1} v_{ij}^{x^j} x^i \in Z(\tilde{W})$, and $v, v' \in V$. Then

$$\begin{aligned} zv &= \sum_{i,j} v_{ij}^{x^j} v^{x^{-i}} x^i = v_{00}v + \sum_{i>0} (v_{i,l^d-i} v^{x^{-l^d}})^{x^{l^d-i}} x^i \in V \oplus \bigoplus_{i=1}^{l^d-1} V^{x^{l^d-i}} x^i, \\ vz &= \sum_{i,j} v v_{ij}^{x^j} x^i = \sum_i v v_{i0} x^i = v v_{00} + \sum_{i>0} v v_{i0} x^i \in V \oplus \bigoplus_{i=1}^{l^d-1} V x^i. \end{aligned}$$

Thus, $v_{00} \in Z(V)$ and the orthogonality of the V^{x^j} implies $v_{i,l^d-i} = 0 = v_{i0}$ for all $i > 0$. Next,

$$\begin{aligned} zv^x &= \sum_{i,j} v_{ij}^{x^j} v^{x^{-i+1}} x^i = (v_{01}v)^x + v_{10}vx + \sum_{i>1} (v_{i,l^d-i+1} v^{x^{-l^d}})^{x^{l^d-i+1}} x^i, \\ v^x z &= \sum_{i,j} v^x v_{ij}^{x^j} x^i = \sum_i (v v_{i1})^x x^i = (v v_{01})^x + (v v_{11})^x x + \sum_{i>1} (v v_{i1})^x x^i. \end{aligned}$$

Thus, $v_{01} \in Z(V)$, $v_{10} = 0 = v_{11}$ and $v_{i,l^d-i+1} = 0 = v_{i1}$ for all $i > 1$. Analogous computations for $zv^{x^\nu} = v^{x^\nu}z$ finally lead to

$$z = v_0 + v_1^x + \dots + v_{l^d-1}^{x^{l^d-1}}$$

with $v_i \in Z(V)$. We apply this together with the orthogonality and compute

$$\begin{aligned} z(v + v'x^i) &= v_0v + v_0v'x, \\ (v + v'x^i)z &= vv_0 + v'v_ix = v_0v + v_iv'x \end{aligned}$$

with $1 \leq i \leq l^d - 1$. Thus, $v_i = v_0$ for all $1 \leq i \leq l^d - 1$. Hence, we have achieved $z \in \{\sum_{j=0}^{l^d-1} v^{x^j} : v \in Z(V)\} \cong Z(V)$, i.e. $Z(\tilde{W}) \subseteq Z(V)$. For the other inclusion, it remains to show that elements of $\{\sum_{j=0}^{l^d-1} v^{x^j} : v \in Z(V)\}$ are already central in \tilde{W} . For this, we only have to check that $\sum_{j=0}^{l^d-1} v^{x^j}$ commutes with x for every $v \in Z(V) = F^{\langle x^{l^d} \rangle}$:

$$\left(\sum_{j=0}^{l^d-1} v^{x^j} \right)^x = \sum_{j=0}^{l^d-1} v^{x^{j+1}} = \sum_{j=1}^{l^d-1} v^{x^j} + v^{x^{l^d}} = \sum_{j=1}^{l^d-1} v^{x^j} + v = \sum_{j=0}^{l^d-1} v^{x^j}.$$

Hence, $Z(\tilde{W}) = Z(V)$ is true. This moreover shows that \tilde{W} is a Wedderburn component of $(\mathcal{Q}_{l(\beta_i)}U_i) \star \langle x \rangle$.

It remains to show that \tilde{W} is the claimed matrix ring. First, if \tilde{W} is a matrix ring over V , then the dimension is clear because

$$\begin{aligned} \dim_{Z(\tilde{W})}\tilde{W} &= \dim_{Z(V)}\tilde{W} = \dim_{Z(V)} \bigoplus_{j=0}^{l^d-1} (V^{x^j} \oplus V^{x^j}x \oplus \dots \oplus V^{x^j}x^{l^d-1}) \\ &= l^{2d} \dim_{Z(V)} V \end{aligned}$$

and therefore $\dim_V \tilde{W} = \dim_{Z(V)} \tilde{W} / \dim_{Z(V)} V = l^{2d}$.

Both V and \tilde{W} are central simple algebras over $F^{\langle y \rangle}$. We show that $V \sim \tilde{W}$ in $\text{Br}(F^{\langle y \rangle})$, i.e. that V and \tilde{W} are full matrix rings over the same skew field D of centre $F^{\langle y \rangle}$. To compute the underlying skew fields, we repeat the strategy we used in the proof of Theorem 3.1 to compute the Wedderburn components considered there.

We recall that for the skew field D in \tilde{W} , there exists a primitive idempotent ε of \tilde{W} such that $D \cong \varepsilon \tilde{W} \varepsilon$ and $\tilde{W} \cong B^n$ for a minimal right ideal $B = \varepsilon \tilde{W}$ of \tilde{W} . Analogously, there exists a primitive idempotent $\varepsilon_V \in V$ with $\varepsilon_V V \varepsilon_V \cong D_V$ a skew field and $S = \varepsilon_V V$ a minimal right ideal of V . Then, we get $\varepsilon_V \tilde{W} \varepsilon_V = \varepsilon_V V \varepsilon_V$

because for $\sum_{i,j=0}^{l^d-1} v_{ij}^{x^j} x^i \in \tilde{W}$, we achieve

$$\begin{aligned} \varepsilon_V \cdot \left(\sum_{i,j=0}^{l^d-1} v_{ij}^{x^j} x^i \right) \cdot \varepsilon_V &= \varepsilon_V \cdot \left(\sum_{i,j=0}^{l^d-1} v_{ij}^{x^j} \varepsilon_V^{-i} x^i \right) \\ &\stackrel{1}{=} \varepsilon_V v_{00} \varepsilon_V + \varepsilon_V \cdot \left(\sum_{i=1}^{l^d-1} (v_{i,l^d-i}^{x^{l^d}})^{x^{-i}} \varepsilon_V^{-i} x^i \right) \\ &= \varepsilon_V v_{00} \varepsilon_V + \left(\sum_{i=1}^{l^d-1} \varepsilon_V (v_{i,l^d-i}^{x^{l^d}})^{x^{-i}} x^i \right) \cdot \varepsilon_V \\ &\stackrel{2}{=} \varepsilon_V v_{00} \varepsilon_V. \end{aligned}$$

For $\stackrel{1}{=}$ and $\stackrel{2}{=}$, we have again used $V^{x^j} \cdot V = 0$ for $1 \leq j \leq l^d - 1$.

Next, as $\varepsilon_V \tilde{W}$ is a right ideal of \tilde{W} , there exists a $0 < r \in \mathbb{N}$ with $B^r \cong \varepsilon_V \cdot \tilde{W}$ for the minimal right ideal B . Because $\text{End}_{\tilde{W}}(B) \cong D$ is the skew field lying in \tilde{W} , we get

$$\begin{aligned} D_V \cong \varepsilon_V V \varepsilon_V &= \varepsilon_V \tilde{W} \varepsilon_V \cong \text{End}_{\tilde{W}}(\varepsilon_V \tilde{W}) \\ &\cong \text{End}_{\tilde{W}}(B^r) \cong \text{End}_{\tilde{W}}(B)_{r \times r} \cong D_{r \times r}. \end{aligned}$$

This enforces $r = 1$ (because, for example, D_V does not have zero divisors whereas $D_{r \times r}$ has for $r > 1$). Thus, the underlying skew fields of V and \tilde{W} are equal, i.e. $V \sim \tilde{W}$ in $\text{Br}(E)$. By $V \subseteq \tilde{W}$, this implies the claim and concludes the proof. \square

Remark 4.1 The \tilde{W} in Proposition 4.3 are all simple components of $(\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i) \star \langle x \rangle$ because $\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i = \bigoplus_W W$ and therefore

$$(\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i) \star \langle x \rangle = \left(\bigoplus_W W \right) \star \langle x \rangle = \bigoplus_{\tilde{W}} \tilde{W}.$$

Corollary 4.1 *With the notations of Proposition 4.3, we have*

$$SK_1(\tilde{W}) = SK_1(V).$$

Proof: This is obvious, as the reduced Whitehead group of a simple algebra does only depend on the underlying skew field but not on the matrix degree. \square

Since the Wedderburn components \tilde{W} are classified now, we can start our study of $SK_1(\tilde{W})$.

Theorem 4.2 *Let $G = \langle s \rangle \rtimes U$ be a \mathbb{Q}_l - l -elementary group with a finite cyclic group $\langle s \rangle$ of order prime to l and U an open pro- l subgroup. Assume that $SK_1(\mathcal{Q}^{N_i} U_i) = 1$ for all i . Then*

$$SK_1(\mathcal{Q}G) = 1.$$

The proof of this theorem will be the subject of the next sections. It depends on the number $l^d = \min\{1 \leq j \leq l^n : W^{x^j} = W\}$. We begin with the extreme cases $d = n$ resp. $d = 0$.

4.1 The case $d = n$

First, let $d = n$, i.e. $W^{x^j} \neq W$ for all $1 \leq j \leq l^n - 1$. Thus,

$$\tilde{W} = \bigoplus_{j=0}^{l^n-1} (W^{x^j} \oplus \dots \oplus W^{x^j} x^{l^n-1}), \quad V = W.$$

Then, Proposition 4.3 implies that $\tilde{W} = V_{l^n \times l^n} = W_{l^n \times l^n}$. Furthermore, both W and \tilde{W} have centre $Z(W) = Z(\tilde{W}) = F$. (Observe that F commutes with x^{l^n} because $x^{l^n} \in U_i$ and $F = Z(W)$ for a Wedderburn component W of $\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i$.) By the above corollary, together with the precondition that $SK_1(\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i) = 1$ and in particular $SK_1(W) = 1$, this also implies

Proposition 4.4 *With the above notations, assume $W^{x^j} \neq W$ for all $1 \leq j \leq l^n - 1$ and $SK_1(\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i) = 1$ for all i . Then*

$$SK_1(\tilde{W}) = SK_1(W) = 1.$$

□

4.2 The case $d = 0$

Next, we consider $d = 0$, i.e. $W^x = W$. Thus,

$$\tilde{W} = \bigoplus_{j=0}^{l^n-1} W x^j, \quad V = \tilde{W}.$$

This time, $Z(\tilde{W}) = F^{\langle x \rangle} = E$ with $[F : E] = l^n$ and $G(F/E) = \langle \sigma \rangle$, where σ is induced by the conjugation by x . Note that

$$\langle \sigma \rangle \cong \langle \bar{x} \rangle \cong \langle \tau \rangle \subseteq G(\mathbb{Q}_l(\beta_i)/\mathbb{Q}_l)$$

with τ as above induced by the action of x on $\mathbb{Q}_l(\beta_i)$.

First, we give a brief outline of the proof of $SK_1(\tilde{W}) = SK_1(V) = 1$.

Step 1 $F \otimes_E V \cong W_{l^n \times l^n} \subseteq V_{l^n \times l^n}$.

Step 2 *There exists a $w \in W_{l^n \times l^n}$ such that the conjugation by $w^{-1}x$ is the automorphism $C_{w^{-1}x} = \sigma \otimes 1$ on $W_{l^n \times l^n}$. It is of order l^n and $(w^{-1}x)^{l^n} \in Z(W_{l^n \times l^n}) = F$.*

Step 3 *$A = (F/E, \sigma, (w^{-1}x)^{l^n}) \subseteq V_{l^n \times l^n}$ is a central simple split E -algebra.*

Step 4 *$Z_{V_{l^n \times l^n}}(A) = (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$.*

Step 5 *$V_{l^n \times l^n} \cong A \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$.*

Step 6 *$SK_1(V) = SK_1((W_{l^n \times l^n})^{\langle w^{-1}x \rangle}) = 1$.*

We now start with the sketched computations. We can read V as free left W -module $V = \bigoplus_{j=0}^{l^n-1} Wx^j$ of rank l^n with basis $1, x, \dots, x^{l^n-1}$. This allows us to formulate

Lemma 4.2 *With the above notations, we have an isomorphism*

$$F \otimes_E V \xrightarrow{\cong} W_{l^n \times l^n} = \text{Hom}_W(V, V), \quad f \otimes v \mapsto l_f \circ r_v,$$

where l_f resp. r_v denotes the left resp. right multiplication with $f \in F$ resp. $v \in V$.

Remark 4.2 In particular, $f \otimes 1 \in F \otimes_E V$ maps to the diagonal matrix $f \cdot \mathbf{1}$ with $\mathbf{1}$ the unit matrix.

Proof: $F \otimes_E V$ and $W_{l^n \times l^n}$ are isomorphic by [28, Cor 7.14], we only have to substitute K by E , A by V and B by F . Then, we get $r = l^n$ and the centralizer $B' = Z_V(F) = W$ implies $F \otimes_E V = F^{op} \otimes_E V \cong W_{l^n \times l^n}$.

We will call the stated isomorphism φ for the moment. We read the actions of the W -endomorphisms of V by the right to ensure that φ is compatible with multiplication. For this, take $f, f' \in F$ and $v, v', a \in V$. Then the commutativity of F yields

$$\begin{aligned} (a)(\varphi(f \otimes v) \circ \varphi(f' \otimes v')) &= (a)((l_f \circ r_v) \circ (l_{f'} \circ r_{v'})) \\ &= f' f a v v' = f f' a v v' \\ &= (a)(l_{ff'} \circ r_{vv'}) = (a)\varphi(ff' \otimes vv') \\ &= (a)\varphi((f \otimes v)(f' \otimes v')). \end{aligned}$$

It now easily follows that φ is a homomorphism of E -algebras.

$F \otimes_E V$ is simple because V is a central simple E -algebra. Thus, $\varphi \neq 0$ implies that φ is injective. By dimension comparison, it is surjective as well. \square

Next, we construct the automorphism $C_{w^{-1}x}$ on $W_{l^n \times l^n}$. On the one hand, conjugation by x is an automorphism c_x on W and can therefore be extended to $W_{l^n \times l^n}$ by letting it act on the matrix entries. Furthermore, we can read x as the diagonal

matrix $M_x = x \cdot \mathbf{1}$ in $V_{l^n \times l^n}$. Then, the extension of c_x on $W_{l^n \times l^n}$ is the conjugation by this matrix M_x . This automorphism on $W_{l^n \times l^n}$ will be called C_x in the sequel and we remark that C_x acts on $F = F \cdot \mathbf{1} = F \otimes_E E$ as σ , with $\langle \sigma \rangle = G(F/E)$ as above.

On the other hand, $\sigma \otimes 1 : F \otimes_E V \rightarrow F \otimes_E V$ is another automorphism on $W_{l^n \times l^n}$. As the restriction of $\sigma \otimes 1$ to $F \otimes_E E$ is by construction the old isomorphism σ , the actions of C_x and $\sigma \otimes 1$ coincide on $F = F \otimes_E E$.

Therefore, $C_x(\sigma \otimes 1)^{-1}$ is a central automorphism on $W_{l^n \times l^n}$, i.e. it acts trivially on the centre $Z(W_{l^n \times l^n}) = F \cdot \mathbf{1} = F$. The theorem of Skolem-Noether now implies that $C_x(\sigma \otimes 1)^{-1}$ is the conjugation C_w by some $w \in W_{l^n \times l^n}$, i.e.

$$\sigma \otimes 1 = C_w^{-1} C_x = C_{w^{-1}x}.$$

As $(\sigma \otimes 1)^{l^n} = \text{id}$, we furthermore conclude $(C_{w^{-1}x})^{l^n} = \text{id}$. This means that the conjugation by $(w^{-1}x)^{l^n} = (w^{-1})^{1+x^{-1}+\dots+x^{-l^n+1}} x^{l^n}$ is trivial on $W_{l^n \times l^n}$ and, as $x^{l^n} \in W$ (more precisely $x^{l^n} \in \mathcal{Q}^{\mathbb{Q}(\beta_i)} U_i$ has a component in W but we suppress this here for the sake of brevity), we conclude

$$(w^{-1}x)^{l^n} \in Z(W_{l^n \times l^n}) = F \cdot \mathbf{1} = F.$$

Finally, we choose

$$A = (F \otimes_E E/E \otimes_E E, \sigma \otimes 1 = C_{w^{-1}x}, (w^{-1}x)^{l^n}) = (F/E, \sigma, (w^{-1}x)^{l^n}).$$

By construction, we have $w \in W_{l^n \times l^n}$ and $x \in V$. Therefore,

$$w^{-1}x \in \bigoplus_{j=0}^{l^n-1} W_{l^n \times l^n} x^j = V_{l^n \times l^n}$$

and hence $A \subseteq V_{l^n \times l^n}$.

Lemma 4.3 *Let $A = (F/E, \sigma, (w^{-1}x)^{l^n})$ be as above. Then A splits, i.e. $A \sim E$ in $\text{Br}(E)$.*

Proof: The cyclic algebra A splits if $(w^{-1}x)^{l^n}$ is a norm element in E , i.e. if there exists an element $f \in F$ with $N_{F/E}(f) = (w^{-1}x)^{l^n}$, where $N_{F/E} = N_{\langle \sigma \rangle}$ is the Galois norm of the field extension F/E . The main ingredient of this proof is a cohomological argument used in the proof of [33, Prop 2].

Let F' be the centre of the Wedderburn component W' of $\mathcal{Q}U_i$ such that

$$\mathbb{Q}_l(\beta_i) \otimes_{\mathbb{Q}_l} W' = W$$

is the examined Wedderburn component of $\mathbb{Q}_l(\beta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i = \mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i$. By Chapter 3, $F' = \mathcal{Q}^{\mathbb{Q}_l(\xi)} \Gamma^{w_x} \cong \text{Quot}(\mathbb{Z}_l[\xi][[T]])$. Next, $R' \cong \mathbb{Z}_l[\xi][[T]]$ is as factorial ring

integrally closed in F' . Then, $F = \mathbb{Q}_l(\beta_i) \otimes_{\mathbb{Q}_l} F'$ and the ring of integers of F is given by $R = \mathbb{Z}_l[\beta_i] \otimes_{\mathbb{Z}_l} R'$.

As $w \in W_{l^n \times l^n}$ and $F = Z(W_{l^n \times l^n})$, conjugation by $(w^{-1}x)$ on F is conjugation by x . Thus, conjugation by $(w^{-1}x)$, resp. by x , acts as σ on F and restricts to the automorphism σ' on F' ; the action of x on $F \cong \mathbb{Q}_l(\beta_i) \otimes_{\mathbb{Q}_l} \text{Quot}(\mathbb{Z}_l[\xi][[T]])$ and $R \cong \mathbb{Z}_l[\beta_i] \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[\xi][[T]]$ is the diagonal action $\sigma = \tau \otimes \sigma'$ induced by x on the factors. Observe that we do not need to know how $(w^{-1}x)$ acts on W and W' but only how it acts on the centres.

Because R' is integrally closed in F' , it is invariant under the action of $\langle \sigma' \rangle$. Therefore, prime elements π in R' map to prime elements $\tilde{\pi}$ under $\langle \sigma' \rangle$. The fundamental theorem on homomorphisms now implies that $R'/(\pi) \cong R'/(\tilde{\pi})$. This in turn means that π and $\tilde{\pi}$ only differ by a unit in R' . Therefore, by choosing $\pi = T$, we achieve that, for $n \in \mathbb{N}$, the $R'/(T^n)$ resp. $T^n R'$ are $\mathbb{Z}_l[\langle \sigma' \rangle]$ -factor resp. submodules of R' .

Next, we pass to $F = \mathbb{Q}_l(\beta_i) \otimes_{\mathbb{Q}_l} F'$ and $R = \mathbb{Z}_l[\beta_i] \otimes_{\mathbb{Z}_l} R'$. Observe that R is invariant under $\langle \sigma \rangle$. Next,

$$R_n := \mathbb{Z}_l[\beta_i] \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[\xi][[T]]/(T^n)$$

for $n \geq 1$ is a $\mathbb{Z}_l[\langle \sigma \rangle]$ -factor module. We use this to show that $R \cong \mathbb{Z}_l[\beta_i] \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[\xi][[T]]$ is a cohomologically trivial $\mathbb{Z}_l[\langle \sigma \rangle]$ -module. R is the projective system $R = \varprojlim R_n$ with $f_n : R_n \rightarrow R_{n-1}$ the n -th projection. Thus, as used in [33, Prop 2], R is cohomologically trivial if R_1 and $\ker(R_n \rightarrow R_{n-1})$ are for all $n > 1$.

We compute $\ker(R_n \rightarrow R_{n-1}) = \ker(f_n)$: For $(f(T) \bmod T^n)$ to lie in $\ker(f_n)$, it is necessary and sufficient that $T^{n-1} \mid f(T)$. Hence,

$$\begin{aligned} \ker(R_n \rightarrow R_{n-1}) &= T^{n-1}(\mathbb{Z}_l[\beta_i] \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[\xi][[T]]/(T^n)) \\ &= \mathbb{Z}_l[\beta_i] \otimes_{\mathbb{Z}_l} T^{n-1}\mathbb{Z}_l[\xi][[T]]/(T^n) \\ &\cong \mathbb{Z}_l[\beta_i] \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[\xi][[T]]/(T) \\ &\cong \mathbb{Z}_l[\beta_i] \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[\xi] \end{aligned}$$

for all $n > 1$ and moreover,

$$\ker(R_n \rightarrow R_{n-1}) \cong R_1.$$

Therefore, we have to show that $\mathbb{Z}_l[\beta_i] \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[\xi]$ is a cohomologically trivial $\mathbb{Z}_l[\langle \sigma \rangle]$ -module. First, $\mathbb{Z}_l[\beta_i]$ is a cohomologically trivial $\mathbb{Z}_l[\langle \sigma \rangle]$ -module because it is unramified over \mathbb{Z}_l as β_i is a primitive root of unity of order prime to l . Moreover, it is torsion free. Then, [42, Cor, p. 145] states that the tensor product $\mathbb{Z}_l[\beta_i] \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[\xi]$ is a cohomologically trivial $\mathbb{Z}_l[\langle \sigma \rangle]$ -module, too. We conclude that R is a cohomologically trivial $\mathbb{Z}_l[\langle \sigma \rangle]$ -module. Thus, we have in particular

$$0 = \hat{H}^0(\langle \sigma \rangle, R) = R^{(\sigma)} / N_{\langle \sigma \rangle} R,$$

where $R^{(\sigma)}$ are the fixed points of R under $\langle \sigma \rangle$ and $N_{\langle \sigma \rangle}$ denotes the norm under the $\langle \sigma \rangle$ -action.

We now turn to the W -component κ of $(w^{-1}x)^{l^n}$. It lies in $E = F^{(\sigma)} = \text{Quot}(R^{(\sigma)})$ and thus $\kappa = z^{-1}\alpha$ with $\alpha, z \in R^{(\sigma)}$, where $R = \mathbb{Z}_l[\beta_i] \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[\xi][[T]]$ as above. Without changing A , we may replace κ by a multiple $\kappa' = \kappa \cdot N_{F/E}(\delta)$ with $\delta \in F^\times$ (see e.g. [28, Thm (30.4)]). Taking $\delta = z$, we arrive at the new

$$\kappa' = z^{[F:E]-1}\alpha \in R^{(\sigma)} = N_{\langle \sigma \rangle} R$$

which shows that A splits. \square

Lemma 4.4 *Set $A = (F/E, \sigma, (w^{-1}x)^{l^n})$ and V as above. Then*

$$Z_{V_{l^n \times l^n}}(A) = (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}.$$

Proof: First, we have to read A in $V_{l^n \times l^n}$. For this, we observe that E and F are to be represented by the diagonal matrices $E = E \cdot \mathbf{1}$ and $F = F \cdot \mathbf{1}$. Then, choose a matrix $(v_{ij})_{i,j} \in Z_{V_{l^n \times l^n}}(A)$ and $f \cdot \mathbf{1} \in F \cdot \mathbf{1}$. We get

$$f^{-1}\mathbf{1}(v_{ij})f\mathbf{1} = (f^{-1}v_{ij}f) \stackrel{!}{=} (v_{ij}),$$

i.e. $f^{-1}v_{ij}f \stackrel{!}{=} v_{ij}$ for all $i, j = 0, \dots, l^n - 1$. As this equation has to be fulfilled for all $f \in F$, but $v_{ij} = w_0 + \dots + w_{l^n-1}x^{l^n-1} \in V$, we conclude that $v_{ij} = w_0 \in W$. Therefore, $Z_{V_{l^n \times l^n}}(A) \subseteq W_{l^n \times l^n}$.

Next, we conjugate by $w^{-1}x$ and see $Z_{V_{l^n \times l^n}}(A) \subseteq (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$.

It remains to show the other inclusion $(W_{l^n \times l^n})^{\langle w^{-1}x \rangle} \subseteq Z_{V_{l^n \times l^n}}(A)$. But this is obvious, because $A = \bigoplus_{j=0}^{l^n-1} F \cdot \mathbf{1}(w^{-1}x)^j$ and $v \in (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ commutes with $w^{-1}x$ as well as with $a \in F \cdot \mathbf{1} = Z(W_{l^n \times l^n})$. Thus, $v \in (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ commutes with $a_0 + a_1w^{-1}x + \dots + a_{l^n-1}(w^{-1}x)^{l^n-1} \in \bigoplus_{j=0}^{l^n-1} F(w^{-1}x)^j = A$, too. \square

Corollary 4.2 $(W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ is a central simple $Z(A) = E$ -algebra.

Proof: This is true by the centralizer theorem. \square

Lemma 4.5 *With the above notations, we have*

$$V \cong (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}.$$

Moreover,

$$F \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle} \xrightarrow{\cong} W_{l^n \times l^n}, \quad f \otimes w \mapsto fw,$$

and

$$A \otimes_E Z_{V_{l^n \times l^n}}(A) \xrightarrow{\cong} V_{l^n \times l^n}, \quad a \otimes v \mapsto av,$$

are isomorphisms.

Proof: First, $V \sim (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ in $\text{Br}(E)$ by the centralizer theorem which states that $Z_{V_{l^n \times l^n}}(A) \sim A^{op} \otimes_E V_{l^n \times l^n}$ in $\text{Br}(E)$. By Lemma 4.3, we know that $A \sim E$ in $\text{Br}(E)$. Because A^{op} is the inverse of A in $\text{Br}(E)$, we conclude $A^{op} \sim E$ in $\text{Br}(E)$, too. Thus,

$$(W_{l^n \times l^n})^{\langle w^{-1}x \rangle} = Z_{V_{l^n \times l^n}}(A) \sim A^{op} \otimes_E V_{l^n \times l^n} \sim E \otimes_E V_{l^n \times l^n} \sim V.$$

Next, we compute the respective degrees over E :

$$[(W_{l^n \times l^n})^{\langle w^{-1}x \rangle} : E] = [W_{l^n \times l^n} : E]/l^n = l^n[W : E] = l^n[W : F][F : E] = l^{2n}[W : F]$$

and

$$[V : E] = [V : W][W : F][F : E] = l^{2n}[W : F].$$

Thus, V and $(W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ are as Brauer equivalent algebras of the same degree isomorphic.

We turn to the second isomorphism. As $Z_{V_{l^n \times l^n}}(A) = (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ is a central simple E -algebra, $F \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ is a central simple F -algebra. We compute the respective degrees over F :

$$[V_{l^n \times l^n} : E] = [V_{l^n \times l^n} : W_{l^n \times l^n}][W_{l^n \times l^n} : F][F : E] = l^{2n}[W_{l^n \times l^n} : F]$$

and, by the centralizer theorem,

$$\begin{aligned} [V_{l^n \times l^n} : E] &= [A : E][Z_{V_{l^n \times l^n}}(A) : E] \\ &= [F : E]^2[(W_{l^n \times l^n})^{\langle w^{-1}x \rangle} : E] = l^{2n}[(W_{l^n \times l^n})^{\langle w^{-1}x \rangle} : E] \end{aligned}$$

imply

$$\begin{aligned} [W_{l^n \times l^n} : F] &= [(W_{l^n \times l^n})^{\langle w^{-1}x \rangle} : E] = [(W_{l^n \times l^n})^{\langle w^{-1}x \rangle} \otimes_E F : E \otimes_E F] \\ &= [(W_{l^n \times l^n})^{\langle x \rangle} \otimes_E F : F]. \end{aligned}$$

Next, $F \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle} \rightarrow W_{l^n \times l^n}$, $f \otimes w \mapsto fw$, is injective because otherwise the kernel would form a non-trivial two-sided ideal. But $F \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ is a central simple F -algebra. Thus, the only non-trivial two-sided ideal is $F \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ itself, which is impossible because $E \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle} \subseteq F \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ maps to $(W_{l^n \times l^n})^{\langle w^{-1}x \rangle} \subseteq W_{l^n \times l^n}$ and thus the kernel can not be $F \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$. This implies $F \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle} \subseteq W_{l^n \times l^n}$. As both sides are of the same degree over F , we conclude $F \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle} = W_{l^n \times l^n}$.

Finally, we show $A \otimes_E Z_{V_{l^n \times l^n}}(A) \cong V_{l^n \times l^n}$. As A and $Z_{V_{l^n \times l^n}}(A)$ are central simple E -algebras, $A \otimes_E Z_{V_{l^n \times l^n}}(A)$ is a central simple E -algebra, too. We again show that the respective degrees over E coincide:

$$[V_{l^n \times l^n} : E] = [A : E][Z_{V_{l^n \times l^n}}(A) : E] = [A \otimes_E Z_{V_{l^n \times l^n}}(A) : E].$$

Next, the homomorphism $A \otimes_E Z_{V_{l^n \times l^n}}(A) \rightarrow V_{l^n \times l^n}$, $a \otimes v \mapsto av$, again is injective. Dimension comparison implies $A \otimes_E Z_{V_{l^n \times l^n}}(A) = V_{l^n \times l^n}$. \square

Proposition 4.5 *With the above notations, assume that $W^x = W$ and moreover $SK_1(\mathcal{Q}^{\mathbb{Q}_l(\beta_i)} U_i) = 1$ for all i . Then*

$$SK_1(\tilde{W}) = SK_1((W_{l^n \times l^n})^{\langle w^{-1}x \rangle}) = 1.$$

Proof: We are still in the case $V = \tilde{W}$. With $V_{l^n \times l^n} = A \otimes_E Z_{V_{l^n \times l^n}}(A)$, it therefore suffices to compute

$$\begin{aligned} SK_1(V) &= SK_1(V_{l^n \times l^n}) = SK_1(A \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}) \\ &\stackrel{1}{=} SK_1(A) \times SK_1((W_{l^n \times l^n})^{\langle w^{-1}x \rangle}) \\ &\stackrel{2}{=} 1 \times SK_1((W_{l^n \times l^n})^{\langle w^{-1}x \rangle}). \end{aligned}$$

in the sequel. For $\stackrel{2}{=}$, we use that A splits and therefore $SK_1(A) = 1$; moreover, A and $(W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ have coprime Schur indices which implies $\stackrel{1}{=}$ by [5, Lem 5, p. 160].

Now, we choose a $v \in V_{l^n \times l^n}$ with $\text{nr}_{V_{l^n \times l^n}/E}(v) = 1$. It represents an element in $SK_1(V_{l^n \times l^n})$. By the above, the class of v can be read as

$$[v] = (1, [\tilde{v}]) = [1 \otimes \tilde{v}] = [\tilde{v}]$$

with $\tilde{v} \in (W_{l^n \times l^n})^{\langle w^{-1}x \rangle} \subseteq V_{l^n \times l^n}$.

Therefore, v and \tilde{v} only differ by a factor in $[(V_{l^n \times l^n})^\times, (V_{l^n \times l^n})^\times]$. It hence suffices to show that $\tilde{v} \in [(V_{l^n \times l^n})^\times, (V_{l^n \times l^n})^\times]$ for $SK_1(V_{l^n \times l^n}) = 1$.

For the computation of $\text{nr}_{(W_{l^n \times l^n})^{\langle w^{-1}x \rangle}/E}$, let M be a splitting field of $(W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ with $M \supseteq F$. Thus, as $(W_{l^n \times l^n})^{\langle w^{-1}x \rangle} \subseteq F \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle} = W_{l^n \times l^n}$,

$$M_{m \times m} = M \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle} = M \otimes_F F \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle} = M \otimes_F W_{l^n \times l^n}$$

for a certain $m \in \mathbb{N}$, i.e. M is also a splitting field of $W_{l^n \times l^n}$. This implies

$$1 = \text{nr}_{Z_{V_{l^n \times l^n}}(A)/E}(\tilde{v}) = \text{nr}_{W_{l^n \times l^n}/F}(\tilde{v}).$$

But, by assumption, $SK_1(W_{l^n \times l^n}) = SK_1(W) = 1$ and hence

$$\tilde{v} \in [(W_{l^n \times l^n})^\times, (W_{l^n \times l^n})^\times] \subseteq [(V_{l^n \times l^n})^\times, (V_{l^n \times l^n})^\times].$$

This concludes the proof. \square

4.3 The intermediate case $0 < d < n$

Finally, the triviality of $SK_1(\tilde{W})$ in the intermediate cases for $0 < j < n$ is a consequence of the extreme cases: We fix a $0 < d < n$ and set $y := x^{l^d}$ and $m := n - d$. Thus,

$$\begin{aligned} V &= W \oplus Wx \oplus \dots \oplus Wx^{l^d(l^{n-d}-1)} \\ &= W \oplus Wy \oplus \dots \oplus Wy^{(l^{n-d}-1)} = \bigoplus_{j=0}^{l^m-1} Wy^j. \end{aligned}$$

As $\tilde{W} = V_{l^d \times l^d}$, it suffices to compute $SK_1(V)$. But V is now of the same form as \tilde{W} in the case $d = 0$, with x replaced by y and n replaced by m . Thus, we only have to check that the above arguments apply to this V in the same manner. As it can be seen easily that we can copy the above literally, we leave this to the reader.

Finally, we have seen that $SK_1(\tilde{W}) = 1$ for every Wedderburn component of $\mathcal{Q}G$. This concludes the proof of Theorem 4.2. \square

4.4 Translation

We now translate these results into the language used in Chapter 1 and 2 when considering the existence claim of the main conjecture of equivariant Iwasawa theory.

Proposition 4.6 *Let $G = \langle s \rangle \rtimes U$ be a \mathbb{Q}_l - l -elementary group. With the above notations, the following is true.*

(i) *With r and r' the natural maps, the diagram*

$$\begin{array}{ccc} K_1(\mathcal{Q}G) & \xrightarrow{r} & \prod_i K_1(\mathcal{Q}^{N_i}U_i) \\ \downarrow \text{Det} & & \downarrow \text{Det} \\ \text{Hom}^*(R_l G, (\mathcal{Q}^c \Gamma_k)^\times) & \xrightarrow{r'} & \prod_i \text{Hom}^{N_i}(R_l U_i, (\mathcal{Q}^c \Gamma_k)^\times) \end{array} \quad (4.2)$$

is commutative.

(ii) *Assume that $SK_1(\mathcal{Q}^{N_i}U_i) = 1$ for all i . Then $SK_1(\mathcal{Q}G) = 1$ if and only if r is injective in the upper diagram.*

Proof: For the commutativity of the diagram read [33, Thm 1] with Λ replaced by \mathcal{Q} .

To prove (ii), we first show $\ker(r) \subseteq SK_1(\mathcal{Q}G)$: Let $x \in \ker(r)$. The commutativity of the diagram means that $r' \circ \text{Det}(x) = \text{Det} \circ r(x) = \text{Det}(1) = 1$. Because r' is

injective, we conclude $\text{Det}(x) = 1$ and thus $x \in SK_1(\mathcal{Q}G)$. Therefore the triviality of $SK_1(\mathcal{Q}G)$ implies the injectivity of r .

On the other hand, we show that $SK_1(\mathcal{Q}G)$ is trivial if r is injective: This time, let $x \in SK_1(\mathcal{Q}G)$. Then $\text{Det} \circ r(x) = r' \circ \text{Det}(x) = r'(1) = 1$ implies that $r(x)$ is in the kernel of the right determinant map, i.e. $r(x) \in \prod_i SK_1(\mathcal{Q}^{N_i}U_i)$ and thus $r(SK_1(\mathcal{Q}G)) \subseteq \prod_i SK_1(\mathcal{Q}^{N_i}U_i)$. By assumption, this product is trivial and thus the injectivity of r implies $SK_1(\mathcal{Q}G) = 1$. \square

Corollary 4.3 *With the notations of Proposition 4.6, assume that $SK_1(\mathcal{Q}^{N_i}U_i) = 1$ for all i . Then the map r is injective.*

If G in fact is a direct product $G = \langle s \rangle \times U$ then we see $U_i = U$ and we are back in the situation of Chapter 2. In this case, diagram (4.2) specializes to the left half of the (uncompleted) diagram (2.1). Particularly, the upper homomorphism r becomes an isomorphism, as seen on Chapter 2.

5 Completion and cohomological dimension

5.1 $\mathcal{Q}_\wedge G$

We again consider pro- l groups G and use our results of Chapter 3 on the structure of $\mathcal{Q}G$ to examine the completed algebra $\mathcal{Q}_\wedge G$.

Let A be the Wedderburn component of $\mathcal{Q}G$ corresponding to the irreducible \mathbb{Q}_l^c -character χ and D its underlying skew field. The centre

$$Z(D) = \mathcal{Q}^L \Gamma^{w_\chi} = L \otimes_{\mathbb{Q}_l} \mathcal{Q} \Gamma^{w_\chi} \cong L \otimes_{\mathbb{Q}_l} \text{Quot}(\mathbb{Z}_l[[T]])$$

is not complete with respect to any \mathfrak{p} -adic valuation where \mathfrak{p} denotes the prime ideal (ℓ) , (T) or $(f(T))$ for an irreducible distinguished polynomial of $\mathfrak{o}_L[[T]]$ (with $\ell \in \mathfrak{o}_L$ of value 1). For example, consider for brevity $L = \mathbb{Q}_l$. Then, the sequence $a_n = \sum_{i=0}^n \frac{l^i}{T^i}$ is Cauchy with respect to the l -adic valuation but does not converge in $\text{Quot}(\mathbb{Z}_l[[T]]) \subsetneq \mathbb{Q}_l((T))$.

For the examination of $\mathcal{Q}_\wedge G$, we will need the concept of higher dimensional local fields.

Definition 5.1 *A complete discrete valuation field Q is called n -dimensional local field if there exists a chain of fields $Q = Q^{(n)}, Q^{(n-1)}, \dots, Q^{(0)}$ such that $Q^{(i+1)}$ is a complete discrete valuation field with residue field $Q^{(i)}$ and $Q^{(0)}$ is a finite field.*

We first cite an example of higher dimensional local fields, the so-called standard fields (see [47, p. 6]). For a complete discrete valuation field F with residue field \overline{F} , we set

$$K_F := F\{\{T\}\} := \left\{ \sum_{i=-\infty}^{\infty} a_i T^i : a_i \in F, \inf\{v_F(a_i)\} > -\infty, \lim_{i \rightarrow -\infty} v_F(a_i) = \infty \right\}.$$

It is a complete discrete valuation field with valuation $v_{K_F}(\sum a_i T^i) = \min\{v_F(a_i)\}$ and residue field $\overline{F}((\overline{T}))$. For a local field F , the fields

$$F\{\{T_1\}\} \dots \{\{T_m\}\}((T_{m+2})) \dots ((T_n)), \quad 0 \leq m \leq n-1,$$

are n -dimensional local fields. They are called standard fields.

Now, we can compute the completions of $\mathcal{Q}^L \Gamma^{w_\chi}$.

Lemma 5.1 *Let X denote a transcendental element over L and $\mathcal{Q}_\wedge^L \Gamma^{w_\chi}$ the completion of $\mathcal{Q}^L \Gamma^{w_\chi}$ with respect to the \mathfrak{p} -adic valuation. Then $\mathcal{Q}_\wedge^L \Gamma^{w_\chi}$ is a two-dimensional local field and*

- (i) $\mathcal{Q}_\wedge^L \Gamma^{w_\chi} \cong L((X))$ if $\mathfrak{p} = (T)$,
- (ii) $\mathcal{Q}_\wedge^L \Gamma^{w_\chi} \cong (L[[T]]/(f))((X))$ if $\mathfrak{p} = (f)$ for an irreducible distinguished polynomial f ,
- (iii) $\mathcal{Q}_\wedge^L \Gamma^{w_\chi} \cong L\{\{T\}\}$ if $\mathfrak{p} = (\ell)$.

Proof: The strategy of the proof is to compute the residue fields and then to choose the right case of the classification theorem in [47] which classifies higher dimensional local fields up to isomorphism.

We set $\Lambda^{\circ_L} \Gamma^{w_\chi} \cong \mathbb{Z}_l[\xi][[T]] =: R$ with ξ a primitive l -power root of unity such that $\mathfrak{o}_L = \mathbb{Z}_l[\xi]$ and $\mathcal{Q}^L \Gamma^{w_\chi} \cong \text{Quot}(R) =: Q$. Furthermore, let $(\cdot)_\bullet$ resp. $(\cdot)_\wedge$ denote the localization with respect to the prime lying above \mathfrak{p} resp. the \mathfrak{p} -adic completion.

In the first case $\mathfrak{p} = (T)$, the valuation ring of $\text{Quot}(\mathbb{Z}_l[\xi][[T]])$ is $\mathfrak{o} := \mathbb{Q}_l(\xi)[[T]]$ and thus we have for the residue field

$$(Q_\wedge)^{(1)} = \mathfrak{o}_\wedge / (T) = \mathfrak{o} / (T) = \mathbb{Q}_l(\xi),$$

which is a local field with (ℓ) -adic valuation. Thus, the next residue field is

$$(Q_\wedge)^{(0)} = \overline{\mathbb{Q}_l(\xi)} = \mathbb{F}_l$$

because $\mathbb{Q}_l(\xi)/\mathbb{Q}_l$ is totally ramified. Therefore, Q_\wedge is a 2-dimensional local field isomorphic to $F((X))$ with F a local field and X a variable. As $(Q_\wedge)^{(1)} = L$, we finally conclude $Q_\wedge \cong L((X))$.

In the case $\mathfrak{p} = (f)$ for a distinguished irreducible polynomial, we see that the valuation ring is $\mathfrak{o} := (\mathbb{Z}_l[\xi][[T]])_\bullet$ and therefore

$$(Q_\wedge)^{(1)} = \mathfrak{o}_\wedge / (f) = \mathfrak{o} / (f) = (\mathbb{Q}_l(\xi)[[T]]) / (f).$$

Because f is as irreducible distinguished polynomial Eisenstein, $(Q_\wedge)^{(1)}$ is totally ramified over \mathbb{Q}_l and thus

$$(Q_\wedge)^{(0)} = \mathbb{F}_l.$$

Hence, we see again that Q_\wedge is a 2-dimensional local field and, moreover, that it is isomorphic to $(L[[T]]/(f))((X))$.

Finally, we show the third isomorphism for the prime $\mathfrak{p} = (\ell)$. Because l is totally ramified in $\mathbb{Q}_l(\xi)$, the prime ideal above l is generated by, say, $\ell = 1 - \xi$. With the valuation ring $\mathfrak{o} = R_\bullet$, we compute

$$(Q_\wedge)^{(1)} = \mathfrak{o}_\wedge / (1 - \xi) = \mathfrak{o} / (1 - \xi) = R_\bullet / (1 - \xi) \supseteq R / (1 - \xi) = \mathbb{F}_l[[\overline{T}]].$$

This is not yet a field but only the ring of integers of the local field $\mathbb{F}_l((\overline{T}))$. Hence, we achieve $(Q_\wedge)^{(1)} \supseteq \mathbb{F}_l((\overline{T}))$. By $(\mathbb{Z}_l[\xi])_\bullet = \mathfrak{o}_L = \mathbb{Z}_l[\xi]$, the coefficient ring of R does not change on passing to the localization R_\bullet and therefore

$$(Q_\wedge)^{(1)} = \mathbb{F}_l((\overline{T})) \quad \text{and} \quad (Q_\wedge)^{(0)} = \mathbb{F}_l[[\overline{T}]] / (\overline{T}) = \mathbb{F}_l.$$

The classification theorem now implies that Q_\wedge is a finite extension of a standard field $F\{\{X\}\}$ with local field F .

For the computation of the exact type of Q_\wedge , we follow the construction in the proof of the classification theorem:

First, $\mathbb{Q}_l\{\{T\}\}$ is a complete discrete valuation field of characteristic 0 and (l) is a prime ideal in $\mathfrak{o}_{\mathbb{Q}_l\{\{T\}\}}$. The valuation is with respect to l , thus its absolute index of ramification is $e(\mathbb{Q}_l\{\{T\}\}) := v_{\mathbb{Q}_l\{\{T\}\}}(l) = 1$. Furthermore, it has the same residue field $\overline{\mathbb{Q}_l\{\{T\}\}} = \mathbb{F}_l((\overline{T}))$ as Q_\wedge . Then [7, II.5.6] implies that Q_\wedge can be viewed as a finite extension of $\mathbb{Q}_l\{\{T\}\}$.

It remains to show that $Q_\wedge = L\{\{T\}\}$. To do so, we compare $LQ_\wedge = Q_\wedge$ and $L\mathbb{Q}_l\{\{T\}\} = L\{\{T\}\}$. For the ramification index, we have

$$e(LQ_\wedge / L\mathbb{Q}_l\{\{T\}\}) = e(Q_\wedge / L\{\{T\}\}) = 1$$

because $1 - \xi$ induces the discrete valuations of the two fields. As both fields are complete discrete valuation fields we conclude

$$[Q_\wedge : L\{\{T\}\}] = n = e(Q_\wedge / L\{\{T\}\})f(Q_\wedge / L\{\{T\}\}) = [\overline{Q_\wedge} : \overline{L\{\{T\}\}}].$$

It therefore follows that the degree of the field extension equals the residue class degree. Finally

$$[Q_\wedge : L\{\{T\}\}] = [\overline{Q_\wedge} : \overline{L\{\{T\}\}}] = [\mathbb{F}_l((\overline{T})) : \mathbb{F}_l((\overline{T}))] = 1$$

implies that $Q_\wedge = L\{\{T\}\}$ is a standard field. \square

In particular, we get that the residue field of $\mathcal{Q}_\wedge^L \Gamma^{w_\chi}$ is a local field, more precisely

Corollary 5.1 *With the notations of Lemma 5.1, we get for the residue fields*

- (i) $\overline{\mathcal{Q}_\wedge^L \Gamma^{w_\chi}} = \overline{\mathcal{Q}^L \Gamma^{w_\chi}} \cong L$ if $\mathfrak{p} = (T)$,
- (ii) $\overline{\mathcal{Q}_\wedge^L \Gamma^{w_\chi}} = \overline{\mathcal{Q}^L \Gamma^{w_\chi}} \cong (L[[T]]/(f))$ if $\mathfrak{p} = (f)$ for an irreducible distinguished polynomial,
- (iii) $\overline{\mathcal{Q}_\wedge^L \Gamma^{w_\chi}} = \overline{\mathcal{Q}^L \Gamma^{w_\chi}} \cong \mathbb{F}_l((\overline{T}))$ if $\mathfrak{p} = (\ell)$.

Proof: We have shown this in the proof of Lemma 5.1. \square

Corollary 5.2 *The cohomological dimension of the completed field is*

$$\text{cd}(\mathcal{Q}_\wedge^L \Gamma^{w_\chi}) = 3.$$

Proof: In the case $\mathfrak{p} \neq (\ell)$, this follows directly from the fact that every local field has cohomological dimension 2 and [25, Thm (6.5.15)].

For $\mathfrak{p} = (\ell)$, the claim is stated in [22, end of §3]¹. \square

Let $D_\wedge = Z(D)_\wedge \otimes_{Z(D)} D$ be the completion of D with respect to the \mathfrak{p} -adic valuation. Then D_\wedge is a central simple algebra over $Z(D_\wedge) = Z(D)_\wedge = \mathcal{Q}_\wedge^L \Gamma^{w_\chi}$ and $[D_\wedge : Z(D_\wedge)] = [D : Z(D)] = s_D^2$ (see [13, Hilfssatz 14.2]).

Proposition 5.1 *Let D_\wedge be as above where the completion is with respect to $\mathfrak{p} \neq (\ell)$. Then*

$$SK_1(D_\wedge) = 1.$$

Proof: Let $\overline{D_\wedge}$ denote the residue skew field of the completed skew field D_\wedge . By the above, $\text{char}(\overline{\mathcal{Q}_\wedge^L \Gamma^{w_\chi}}) = 0$ and therefore $\overline{\mathcal{Q}_\wedge^L \Gamma^{w_\chi}}$ is perfect. In particular, the field extension $Z(\overline{D_\wedge})/\overline{\mathcal{Q}_\wedge^L \Gamma^{w_\chi}}$ is separable. Moreover, we have seen that $\overline{\mathcal{Q}_\wedge^L \Gamma^{w_\chi}}$ is a local field. Then the proposition follows readily from Korollar 7 in [4]. We only have to substitute k by $\mathcal{Q}_\wedge^L \Gamma^{w_\chi}$ and Draxl's skew field D is our D_\wedge . Observe that local fields are reasonable (vernünftig). \square

Now, we consider the case when $\text{char}(\overline{\mathcal{Q}_\wedge^L \Gamma^{w_\chi}}) = \ell$, i.e. the completion is with respect to the (ℓ) -adic valuation. Then, $\overline{\mathcal{Q}_\wedge^L \Gamma^{w_\chi}} \cong \mathbb{F}_l((\overline{T}))$ is not perfect. Thus, $Z(\overline{D_\wedge})/\overline{\mathcal{Q}_\wedge^L \Gamma^{w_\chi}}$ might not be separable and we are no longer in the situation of Draxl's Korollar 7. Yet, the following results can be transferred to this situation.

Proposition 5.2 *Let D_\wedge be as above where the completion is with respect to $\mathfrak{p} = (\ell)$. Then D_\wedge is a skew field with Schur index $s_{D_\wedge} = s_D$ and its residue skew field $\overline{D_\wedge}$ is commutative.*

¹A. Weiss has drawn my attention to this paper.

Proof: Let $M = \mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma^{w_\chi} = \mathcal{Q}^{\mathbb{Q}_l(\zeta)}\Gamma^{w_\chi}$ be the maximal subfield of

$$D = (\mathcal{Q}^{\mathbb{Q}_l(\eta)}\Gamma^{w_\chi} / \mathcal{Q}^L\Gamma^{w_\chi}, \sigma_{v_\chi}, \gamma^{w_\chi}),$$

i.e. $s_D = [D : M] = [M : Z(D)]$. The prime ideal $\mathfrak{p} = (\ell)$ in $Z(D) = \mathcal{Q}^L\Gamma^{w_\chi}$ is generated by $(1 - \xi)$. In $M/Z(D)$, this prime ideal is totally ramified and undecomposed. Therefore, $M \otimes_{Z(D)} Z(D)_\wedge = M_\wedge$ is a field of degree $[M_\wedge : Z(D)_\wedge] = s_D$. M_\wedge is thus a maximal subfield in D_\wedge because

$$Z(D)_\wedge \stackrel{s_D}{\subseteq} M \otimes_{Z(D)} Z(D)_\wedge \stackrel{s_D}{\subseteq} D \otimes_{Z(D)} Z(D)_\wedge = D_\wedge.$$

In particular, we conclude

$$D_\wedge = (M_\wedge / Z(D)_\wedge, \sigma_{v_\chi}, \gamma^{w_\chi}).$$

Next, we show that this crossed product has Schur index s_D . We again have to check that the order $o(\gamma^{w_\chi})$ in

$$Z(D_\wedge)^\times / N_{M_\wedge / (Z(D)_\wedge)}(M_\wedge)^\times =: Z(D_\wedge)^\times / N(M_\wedge)^\times$$

is exactly s_D . Again, $o(\gamma^{w_\chi})$ divides $s_D = l^r$ because of $N(\gamma^{w_\chi}) = (\gamma^{w_\chi})^{s_D}$.

Now, we assume that $o(\gamma^{w_\chi}) = l^t$ with $t < r$. Then, there exists an $a \in M_\wedge$ such that $N(a) = (\gamma^{w_\chi})^{l^t}$. Therefore, a is integral as γ^{w_χ} is. Furthermore, the residue fields of M_\wedge and $Z(D)_\wedge$ coincide as \mathfrak{p} is totally ramified, i.e. $G(\overline{M_\wedge} / \overline{Z(D)_\wedge}) = \langle \overline{\sigma_{v_\chi}} \rangle = 1$. We achieve

$$(\overline{\gamma^{w_\chi}})^{l^t} = \overline{N(a)} = \prod_{j=0}^{s_D-1} \overline{a^{\sigma_{v_\chi}^j}} = \overline{a}^{s_D} = \overline{a}^{l^r}.$$

If, as we assume, $t < r$, then $\overline{\gamma^{w_\chi}}$ is an l -th power. To show that this is not possible, we use the isomorphisms $\mathcal{Q}_\wedge^L\Gamma^{w_\chi} \cong L\{\{T\}\}$ of Lemma 5.1 and $\overline{\mathcal{Q}_\wedge^L\Gamma^{w_\chi}} \cong \mathbb{F}_l((\overline{T}))$ of Corollary 5.1. Indeed,

$$\overline{\gamma^{w_\chi}} \leftrightarrow 1 + \overline{T} = \left(\sum_{i=-n}^{\infty} \alpha_i \overline{T}^i \right)^l = \sum_{i=-n}^{\infty} \alpha_i^l \overline{T}^{il}$$

is a contradiction.

Finally, we show that $\overline{D_\wedge}$ is commutative. We use valuation theory for the skew field D_\wedge , for details see e.g. [4, §3].

We start with computing $[\overline{D_\wedge} : \overline{Z(D)_\wedge}]$. For this, let v denote the \mathfrak{p} -adic valuation of $Z(D_\wedge)$ induced by l , i.e. $\mathfrak{p} = (1 - \xi)$. The extension w of v to D_\wedge is defined by

$$w(d) = \frac{1}{s_{D_\wedge}} v(\text{nr}_{D_\wedge / Z(D_\wedge)}(d))$$

for every $d \in D_\wedge$. We then compute the ramification index

$$e(D_\wedge/Z(D_\wedge)) = [w(D_\wedge^\times) : v(Z(D_\wedge)^\times)].$$

The definition of w implies that $e(D/Z(D)) \leq s_{D_\wedge}$. As $\mathbb{Q}_l(\eta) = \mathbb{Q}_l(\zeta)$ is totally ramified over $L = \mathbb{Q}_l(\xi)$, we have

$$N_{\mathbb{Q}_l(\zeta)/\mathbb{Q}_l(\xi)}(1 - \zeta) = \nu \cdot (1 - \xi)$$

for a unit ν in the valuation ring of $\mathbb{Q}_l(\xi)$.

We now choose $d = 1 - \zeta \in M_\wedge^\times \subseteq D_\wedge^\times$ and compute

$$\begin{aligned} w(1 - \zeta) &= \frac{1}{s_{D_\wedge}} v(\text{nr}_{D_\wedge/Z(D_\wedge)}(1 - \zeta)) = \frac{1}{s_{D_\wedge}} v(N_{M_\wedge/Z(D_\wedge)}(1 - \zeta)) \\ &= \frac{1}{s_{D_\wedge}} v(N_{\mathbb{Q}_l(\zeta)/\mathbb{Q}_l(\xi)}(1 - \zeta)) = \frac{1}{s_{D_\wedge}} v(\nu(1 - \xi)) = \frac{1}{s_{D_\wedge}}. \end{aligned}$$

This implies that $e(D_\wedge/Z(D_\wedge)) \geq s_{D_\wedge}$ and finally $e(D_\wedge/Z(D_\wedge)) = s_{D_\wedge}$. By the equation

$$s_{D_\wedge}^2 = [D_\wedge : Z(D_\wedge)] = e(D_\wedge/Z(D_\wedge)) [\overline{D_\wedge} : \overline{Z(D_\wedge)}] = s_{D_\wedge} [\overline{D_\wedge} : \overline{Z(D_\wedge)}],$$

we achieve that

$$[\overline{D_\wedge} : \overline{Z(D_\wedge)}] = s_{D_\wedge}.$$

Next, we consider $N_\wedge := Z(D_\wedge)(\gamma^{v_\chi})$. We have seen that $Z(D_\wedge) \cong L\{\{T\}\}$ with $\gamma^{w_\chi} \leftrightarrow 1+T$. As γ^{v_χ} commutes with L and γ^{w_χ} , this implies that N_\wedge is commutative. Moreover, $N_\wedge \cong L\{\{T\}\}[X]/(f(X))$ with $f(X) = X^{w_\chi/v_\chi} - (1+T) = X^{s_{D_\wedge}} - (1+T)$. The polynomial $f(X)$ is irreducible modulo \mathfrak{p} and thus $f(X)$ is irreducible itself. Thus, N_\wedge is a subfield $Z(D_\wedge) \subseteq N_\wedge \subseteq D_\wedge$ with

$$[\overline{N_\wedge} : \overline{Z(D_\wedge)}] = [N_\wedge : Z(D_\wedge)] = s_{D_\wedge} = [\overline{D_\wedge} : \overline{Z(D_\wedge)}].$$

We finally conclude that $\overline{D_\wedge} = \overline{N_\wedge}$ is commutative. \square

Our interest in $\mathcal{Q}_\wedge G$ and $\mathcal{Q}_\wedge^L \Gamma^{w_\chi}$ arises from Suslin's conjecture which, when true, would give the uniqueness property of (MC). Since fields of type $\mathcal{Q}^L \Gamma^{w_\chi}$ have cohomological dimension 3, as we are going to show in the next section, using a geometrical result of S. Saito, there is hope for $SK_1(\mathcal{Q}G)$ to be trivial. For the completed fields $\mathcal{Q}_\wedge^L \Gamma^{w_\chi}$, we have seen here that they are of the right cohomological dimension. As these are the centres of the Wedderburn components of $\mathcal{Q}_\wedge G$, we even have proven

Corollary 5.3 *Let $\mathcal{Q}_\wedge G$ be the Iwasawa algebra completed with respect to the prime (T) or $(f(T))$ for an irreducible distinguished polynomial $f(T)$. Then*

$$SK_1(\mathcal{Q}_\wedge G) = 1.$$

5.2 Adapting Saito's result

Unfortunately, we are not able to prove the triviality of $SK_1(\mathcal{Q}G)$ at the moment. But we show that the centres of the Wedderburn components of $\mathcal{Q}G$ are of cohomological dimension 3. This reduces the question to the conjecture by Suslin which says that $SK_1(D) = 1$ if $Z(D)$ is of cohomological dimension 3. This also justifies the remark in [32, p. 565] that the centre fields in $Z(\mathcal{Q}G)$ have cohomological dimension 3 by a result of Kato (which was published by S. Saito in [40]).

Theorem 5.1 *Let D be the underlying skew field of a simple component of $\mathcal{Q}G$. Then*

$$\text{cd}(Z(D)) = 3.$$

For the proof of the theorem, we essentially need Theorem 5.1 of [40]:

Theorem 5.2 (S. Saito) *Let A be a 2-dimensional excellent normal henselian local ring, K_A its field of fractions and F_A its algebraically closed residue field. Then*

$$\text{cd}_p(K_A) = 2 \text{ for every prime number } p \neq \text{char}(K_A).$$

Proof of Theorem 5.1: We have $Z(D) = \mathcal{Q}^L \Gamma^{w_\chi} = \mathcal{Q}^{\mathbb{Q}_l(\xi)} \Gamma^{w_\chi}$ for a primitive l -power root of unity ξ . In the completed situation, we have seen that the cohomological dimension is 3. Thus, in the uncompleted case, the cohomological dimension is at least 3 by the following: For any field E , let $\text{Gal}(E)$ denote its absolute Galois group. Then, $\text{Gal}(\mathcal{Q}_\wedge^L \Gamma^{w_\chi})$ is a closed subgroup of $\text{Gal}(\mathcal{Q}^L \Gamma^{w_\chi})$ because it is the decomposition group of some prime above \mathfrak{p} , where the completion is with respect to \mathfrak{p} . Thus, [25, (3.3.5)] shows the claim:

$$3 = \text{cd}(\mathcal{Q}_\wedge^L \Gamma^{w_\chi}) \leq \text{cd}(\mathcal{Q}^L \Gamma^{w_\chi}).$$

Furthermore, we see that $\text{cd}(\mathcal{Q}^{\mathbb{Q}_l(\xi)} \Gamma^{w_\chi}) \leq \text{cd}(\mathcal{Q} \Gamma^{w_\chi})$ again by [25, (3.3.5)] because $\text{Gal}(\mathcal{Q}^{\mathbb{Q}_l(\xi)} \Gamma^{w_\chi})$ is a closed subgroup of $\text{Gal}(\mathcal{Q} \Gamma^{w_\chi})$.

We thus consider the problem whether $\text{cd}(K_A) = 3$ for $K_A = \mathcal{Q} \Gamma^{w_\chi}$ and $A = \Lambda \Gamma^{w_\chi}$. Then, $\text{cd}(\mathcal{Q}^L \Gamma^{w_\chi}) \leq 3$ follows.

The residue field $F_A = A/\mathfrak{m} \cong \mathbb{Z}_l[[T]]/(l, T) = \mathbb{F}_l$ is not algebraically closed and we thus can not apply Saito's theorem directly. To achieve the algebraic closure, we replace A by the ring $A' = \varinjlim \mathbb{Z}_l[\zeta_i][[T]] = \bigcup \mathbb{Z}_l[\zeta_i][[T]]$ with $(l, i) = 1$ and ζ_i a primitive root of order i .

First, we look more closely to A' . Set $I = \mathbb{N} \setminus \mathbb{N}$ the set of natural numbers coprime to l . This is filtered with the relation $i \leq j : \Leftrightarrow i \mid j$. We set

$$A_i = \mathbb{Z}_l[\zeta_i][[T]] = \mathbb{Z}_l[[T]][\zeta_i]$$

and define for $i \leq j$ the inclusion $\varphi_{ij} : A_i \rightarrow A_j$. Then, $A' := \varinjlim A_i = \bigcup A_i$ is the filtered inductive limit of (A_i, φ_{ij}) .

Moreover, A' is locale-ind-étale in the sense of [27, p. 80] by the following: locale-ind-étale means that A' is the filtered inductive limit of some (A_i, φ_{ij}) , where the φ_{ij} are local morphisms and the A_i are locale-étale A -algebras. First, A_i is local with maximal ideal $\mathfrak{m}_i = (l, T)$ because l and i are coprime. Next, $\varphi_{ij} : A_i \rightarrow A_j$ is local, i.e. $\varphi_{ij}^{-1}(\mathfrak{m}_j) \subseteq \mathfrak{m}_i$, because \mathfrak{m}_i and \mathfrak{m}_j are both generated by l and T . A_i is said to be locale-étale over A if it is the localization $B_{\mathfrak{n}}$ of an étale A -algebra B by a prime ideal \mathfrak{n} over \mathfrak{m} . We choose $B = A_i$ and $\mathfrak{n} = \mathfrak{m}_i = (l, T)$. Then $B_{\mathfrak{n}} = B = A_i$. Thus, we only have to show that A_i is étale over A . To do so, we use [10, (18.4.5)]. A and A_i are local rings. Moreover, $A_i = A[\zeta_i] = A[X]/(F(X))$, with $F(X)$ an irreducible polynomial dividing the i -th cyclotomic polynomial, is a finite A -algebra of finite presentation. F is unitary and separable in the meaning of [10, p. 118], i.e. $F'(\zeta_i) \notin \mathfrak{m}_i = (l, T)$ for the generator ζ_i of A_i over A with minimal polynomial F , which can be seen as follows: $F'(\zeta_i) \notin \mathfrak{m}_i = (l, T)$ is equivalent to $F'(\zeta_i) \not\equiv 0 \pmod{\mathfrak{m}_i}$ or $\overline{F'(\zeta_i)} \neq 0$ in $A_i/\mathfrak{m}_i = \mathbb{F}_l(\overline{\zeta_i}) = \mathbb{Z}_l[\zeta_i]/(l)$. Because $\mathbb{Z}_l[\zeta_i]$ is unramified over \mathbb{Z}_l , we know that \overline{F} is also the minimal polynomial of $\overline{\zeta_i}$ over \mathbb{F}_l and in particular irreducible and separable. Thus, $\overline{F'(\zeta_i)} \neq 0$ in $\mathbb{F}_l(\overline{\zeta_i})$ and hence F is a unitary separable polynomial as needed. Hence, [10, (18.4.5)] implies that A_i is an étale A -algebra.

Now, we have $K_{A'} = \text{Quot}(A') = \mathbb{Q}_l^{\text{nr}} \otimes_{\mathbb{Q}_l} K_A$ and

$$G(K_{A'}/K_A) \cong G(\mathbb{Q}_l^{\text{nr}}/\mathbb{Q}_l) = \hat{\mathbb{Z}}.$$

Assume for the moment that A' fulfils the conditions of Saito's theorem and thus $\text{cd}_p(K_{A'}) = 2$ for all $p \neq \text{char}(K_{A'}) = 0$, i.e. $\text{cd}(K_{A'}) = 2$.

$\text{Gal}(K_{A'})$ is a closed normal subgroup of $\text{Gal}(K_A)$, therefore $\text{cd}(\hat{\mathbb{Z}}) = 1$ (see [25, p. 140]) and [25, (3.3.7)] imply

$$\begin{aligned} \text{cd}(K_A) &\leq \text{cd}(K_{A'}) + \text{cd}(\text{Gal}(K_A)/\text{Gal}(K_{A'})) \\ &= \text{cd}(K_{A'}) + \text{cd}(G(K_{A'}/K_A)) = \text{cd}(K_{A'}) + \text{cd}(\hat{\mathbb{Z}}) = 3. \end{aligned}$$

Now, we show that A' fits into the situation of Saito. First, we observe that by construction of A' , the residue field $F_{A'}$ of A' is the separable closure of $F_A = \mathbb{F}_l$. Moreover, F_A is perfect as a finite field and thus its separable closure $F_{A'}$ is already algebraically closed, i.e. $F_{A'} = \mathbb{F}_l^c$.

Next, we show that A' is local. We use [27, Prop 1, p. 6] which says that the filtered inductive limit A' is local because the φ_{ij} are local morphisms and the A_i local rings. Moreover, the proof of [27, Prop 1, p. 6] shows that the maximal ideal of A' is $\mathfrak{m}' = (l, T)$.

Analogously, we show that A' is henselian. The A_i are henselian because they are local and complete in the \mathfrak{m}_i -adic topology (see [25, p. 242]). Thus, A' is henselian again by [27, Prop 1, p. 6].

For A' to be normal, we show that every integral quotient $\frac{a}{b} \in K_{A'}$ lies in A' . There exists a finite subextension $A \subseteq A_i \subsetneq A'$ with $a, b \in A_i$. Then, A_i is normal as finite extension of the normal ring A . Hence, we conclude $\frac{a}{b} \in A_i$ and thus $\frac{a}{b} \in A'$.

The Krull dimension of A' is 2 as needed: Let $(l, T) \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_2$ be a descending chain of prime ideals in A' . We cut down this chain to every $\mathbb{Z}_l[\zeta_i][[T]] = A_i$. Because the maximal ideal of this $\mathbb{Z}_l[\zeta_i][[T]]$ is also generated by l and T , the section of (l, T) remains the maximal ideal and $A_i \cap (l, T) \supsetneq A_i \cap \mathfrak{p}_1$ (otherwise $\mathfrak{p}_1 = (l, T)$, a contradiction). We show that $\mathfrak{p}_2 = 0$. For this, assume that $\mathfrak{p}_2 \neq 0$, i.e. there exists an element $0 \neq a \in \mathfrak{p}_2$ and a certain ζ_i such that $a \in \mathbb{Z}_l[\zeta_i][[T]]$. But $\mathbb{Z}_l[\zeta_i][[T]]$ has dimension 2 and therefore \mathfrak{p}_1 and \mathfrak{p}_2 coincide for $\mathbb{Z}_l[\zeta_i][[T]]$. Thus, \mathfrak{p}_1 and \mathfrak{p}_2 coincide for all $\mathbb{Z}_l[\zeta_j][[T]]$ with ζ_j a primitive root of unity of order prime to l such that $\zeta_j \mid \zeta_i$. Finally, $\mathfrak{p}_1 = \mathfrak{p}_2$ in A' because these $\mathbb{Z}_l[\zeta_j][[T]]$ already generate A' .

It remains to show that A' is excellent. First, we check that A' is Noetherian. For this, we use [27, Thm 3.3, p. 94] which says that the locale-ind-étale A -algebra A' is Noetherian if and only if A is Noetherian. But $A = \mathbb{Z}_l[[T]]$ is Noetherian because it is a power series ring over the Noetherian ring \mathbb{Z}_l by [18, Thm 9.4, p. 210]. Next, A is a Noetherian local ring and complete in its \mathfrak{m} -adic topology (see [25, p. 242]). Thus, A is excellent by [20, p. 260]. Then, [9, Thm 5.3.iv]² implies that A' is also excellent because the inclusion $A \rightarrow A'$ is ind-étale and thus AF, compare [9, Def 5.1.].

This concludes the proof of the theorem. □

Corollary 5.4 *For the group $\Gamma \cong \mathbb{Z}_l$ we have*

$$\mathrm{cd}(\mathcal{Q}\Gamma) = 3.$$

²M. Nieper-Wißkirchen has drawn my attention to this paper.

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Lebenslauf

Persönliche Daten:

Name	Irene Lau, geb. Sommer
Geburtsdaten	08.07.1983 in Ulm

Schulbildung:

1993-2002	Besuch des Bertha-von-Suttner-Gymnasiums Neu-Ulm Abschluss: Abitur (2002)
-----------	------------------------------------------------------------------------------

Studium:

2002-2007	Studium der Mathematik und Physik an der Universität Augsburg Abschlüsse: Diplom in Mathematik (2006), Erste Staatsprüfung für das Lehramt an Gymnasien in Mathematik und Physik (2007)
-----------	-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

Berufstätigkeit:

2007-2009	Wissenschaftliche Mitarbeiterin an der Universität Augsburg
-----------	-------------------------------------------------------------

Stipendien:

2009-2010	Stipendiatin der Studienstiftung des deutschen Volkes
-----------	-------------------------------------------------------